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# THE INTEGRAL SOLUTION OF THE SOUND FIELD IN A MULTILAYERED LIQUID-SOLID HALF-SPACE WITH NUMERICAL COMPUTATIONS FOR LOW-FREQUENCY PROPAGATION IN THE ARCTIC OCEAN

by

Henry W. Kutschale

CU-1-70

Technical Report No. 1

Contract N00014-67-A-0108-0016 with the
Office of Naval Research

Work done on behalf of the U.S. Naval Ordnance Laboratory,
White Oak, Silver Spring, Maryland

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Corrections

Page 7, line 3, should read as follows:

in terms of elements of

Page 13, line 4, should read as follows:

$$(p_{zz})_{n-1} = A_{21}\,\hat{\dot{w}}_0 + A_{22}(\hat{p}_{zz})_0$$

Page 16, line 8, should read as follows:

$$\begin{bmatrix} \hat{\dot{w}}_{n-1} \\ (p_{zz})_{n-1} \end{bmatrix} = a_{n-1} \cdots b_{n-3} a_{n-4} \cdots b_1 \begin{bmatrix} \hat{\dot{w}}_0 \\ 0 \end{bmatrix}$$

Page 19, line 1, should read as follows:

$$A = A_{S_2} A_{S_1} = a_{n-1} a_{n-2} \cdots b_{n-10} \cdots a_1$$

Page 22, line 7, should read as follows:

at depth z at the bottom of the $D_1$ layer is $p_{D_1}(r, z, k, t) =$

Page 22, line 10, should read as follows:

$$A_{D_1} = a_{D_1} a_{D_1 - 1} \cdots a_1$$

Page 28, line 2, should read as follows: but from the relation

$$AA^{-1} = I \quad , A_{11}A_{22} - A_{12}A_{21} = 1$$

Page 41, second line, should read as follows:

$$\left\{ \frac{2\sqrt{2\pi}\sqrt{\omega}}{\rho_n c^{5/2} \bar{A}_{11} \frac{\partial}{\partial c}\left[\frac{c^2 \bar{A}_{11}}{Im(\Gamma_{\alpha n})} + \frac{Im(-\bar{A}_{21})}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_m \\ c=c_m}}$$

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## ABSTRACT

This report develops by matrix methods the integral solution of the wave equation for point sources of harmonic waves in a liquid layer of a multilayered liquid-solid half space in a form convenient for numerical computation on a high-speed digital computer. Only the case is considered of a high-speed liquid bottom underlying the stack of layers. The integral over wave number has singularities in the integrand and is conveniently transformed into the complex plane. By a proper choice of contours, complex poles are displaced to an unused sheet of the two-leaved Riemann surface, and the integral solution for the multilayered system reduces to a sum of normal modes plus the sum of two integrals, one along the real axis and the other along the imaginary axis. Both integrals are evaluated by a Gaussian quadrature formula. Sample computations are presented for low-frequency propagation in the Arctic Ocean sound channel. These are preliminary computations and the ice layer, which averages three meters in thickness, is not included in the layered system. The effects of the ice layer on propagation are currently under investigation.

FIGURES

DEFINITION OF SYMBOLS

$\alpha_m$; compressional-wave velocity in the m-th layer

$\beta_m$; shear-wave velocity in the m-th layer

$h_m$; thickness of the m-th layer

$z$ ; vertical coordinate

$r$ ; range between source and detector

$t$ ; time

$\omega$ ; angular frequency

$c$ ; phase velocity

$k$ ; wave number

$i$ ; $\sqrt{-1}$

$I_m$; imaginary part of a complex number x + iy

$\phi_m$; velocity potential in the m-th layer

$(p_{zz})_m$ ; normal stress in the m-th layer parallel to z axis

$p_m$ ; pressure in the m-th liquid layer

$u_m$; horizontal particle displacement in the m-th layer

$w_m$; vertical particle displacement in the m-th layer

$\dot{u}_m$; horizontal particle velocity in the m-th layer

$\dot{w}_m$; vertical particle velocity in the m-th layer

$\rho_m$; density in the m-th layer

$\rho_s$ ; density at the source

$k_{\alpha_m} = \omega/\alpha_m$

$k_{\beta_m} = \omega/\beta_m$

$J_0$ ; Bessel function of order 0

$Y_0$ ; Y bessel function of order 0

$K_0$ ; K Bessel function of order 0

$H_0^{(1)}$ ; Hankel function of the first kind of order 0

$H_0^{(2)}$; Hankel function of the second kind of order 0

$$\beta_{\alpha m} = \sqrt{k_{\alpha m}^2 - k^2} \quad , \quad k < k_{\alpha m}$$

$$\beta_{\alpha m} = -i\sqrt{k^2 - k_{\alpha m}^2} \quad , \quad k > k_{\alpha m}$$

$$\beta_{\beta m} = \sqrt{k_{\beta m}^2 - k^2} \quad , \quad k < k_{\beta m}$$

$$\beta_{\beta m} = -i\sqrt{k^2 - k_{\beta m}^2} \quad , \quad k > k_{\beta m}$$

$$P_m = h_m \beta_{\alpha m}$$

$$Q_m = h_m \beta_{\beta m}$$

$$r_{\alpha m} = \sqrt{\left(\frac{c}{\alpha_m}\right)^2 - 1} \quad , \quad c > \alpha_m$$

$$r_{\alpha m} = -i\sqrt{1 - \left(\frac{c}{\alpha_m}\right)^2} \quad , \quad c < \alpha_m$$

$$r_{\beta m} = \sqrt{\left(\frac{c}{\beta_m}\right)^2 - 1} \quad , \quad c > \beta_m$$

$$r_{\beta m} = -i\sqrt{1 - \left(\frac{c}{\beta_m}\right)^2} \quad , \quad c < \beta_m$$

$$\bar{P}_m = k h_m r_{\alpha_m}$$

$$\bar{Q}_m = k h_m r_{\beta_m}$$

$$\gamma_m = 2\left(\frac{\beta_m}{c}\right)^2$$

## INTRODUCTION

This report develops the integral solution of the wave equation for point sources of harmonic waves in a liquid layer of a multilayered liquid-solid half space in a form convenient for numerical computation on a high-speed digital computer. Only the case of a liquid half-space underlying the stack of layers is considered. Furthermore, it is assumed that the speed of sound in this last layer of infinite thickness is greater than the speed of sound in liquid layers immediately above it. The problem of immediate concern is low-frequency propagation from harmonic sources in the central Arctic Ocean at ranges up to ten times the water depth in shallow and intermediate water depths. Solutions are developed for the pressure perturbations detected by a hydrophone at depth or the ice vibrations detected by a seismometer on the ice surface. A modification of the formulation is possible to include harmonic vibration sources on or in the ice, although it appears at the present time that surface sources are of limited application (Greene, 1968).

The solution of the wave equation presented here, based on the Thomson-Haskell matrix method (Thomson, 1950; Haskell, 1953), follows Harkrider (1964) for the solution of the wave equation in an n-layered solid half space. Layer matrices of the type given by Dorman (1962) for computing dispersion in an n-layered liquid - solid half-space are used for the liquid layers. An application of the theorem that the inverse of the

product matrix for the layered system above the source has the same form as the inverse of a layer matrix reduces the integrand of the integral solution to a simple form in terms of product matrices derived in the source-free, plane-wave case. The integral over wave number from zero to infinity has singularities in the integrand and is, therefore, conveniently transformed into the complex $\zeta = k + i\tau$ plane. Performing the contour integration in the $\zeta$ plane and noting that branch line integrals corresponding to branch points generated by each layer matrix except the last are zero, we are left with an expression analogous to that of Sorensen (1959) and Leslie and Sorensen (1961) for the two-layer liquid half-space with a high-speed bottom. By the proper choice of contours, complex poles are displaced to an unused sheet of the two-leaved Riemann surface, and the integral solution for the multilayered system reduces to a summation of normal modes plus the sum of two integrals over wave number, one along the real $\zeta$ axis and the other along the imaginary $\zeta$ axis. Both of these integrals are conveniently computed by a Gaussian quadrature formula.

The physical interpretation of the final solution is straightforward. The normal-mode terms correspond to waves trapped in the Arctic sound channel; that is, refracted and surface reflected (RSR) sounds and reflected sounds incident on the bottom beyond the critical angle. The integral over wave number along the real axis corresponds to waves incident

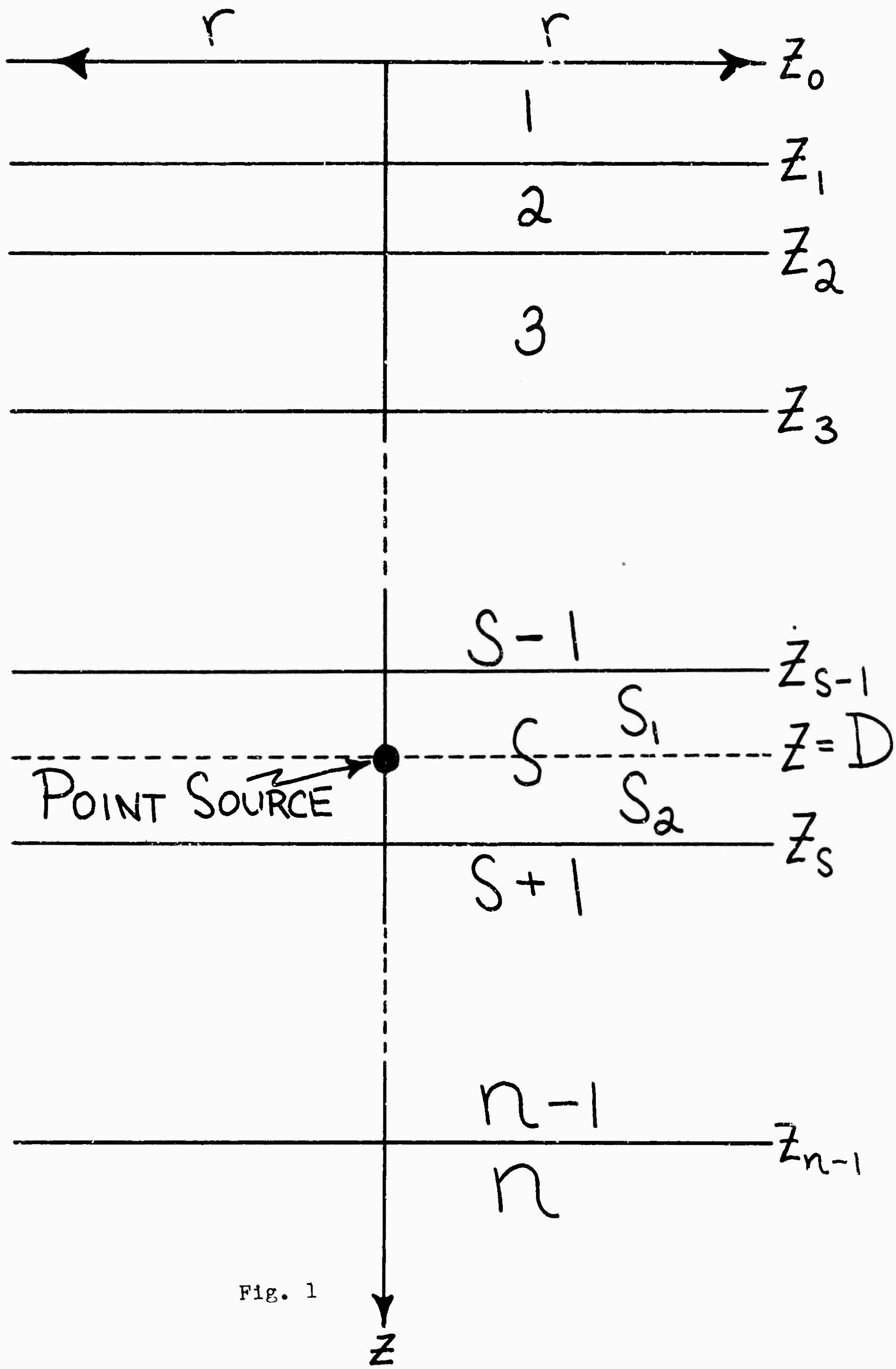


Fig. 1

on the bottom at angles greater than the critical angle of total reflection. Since energy leaves the guide continuously as waves travel down the guide, this term is of importance only to ranges of about ten times the water depth. The contribution to the sound field of the integral along the imaginary axis is of importance only very near the source since the integrand decays exponentially with range and wave number.

The usefulness of the normal mode terms for describing long-range explosive sound transmission in the central Arctic Ocean is shown by Kutschale (1969). In that work the present formulas for the normal modes were extended to explosive sources and the Fourier integral for each mode was evaluated by the principle of stationary phase (Pekeris, 1948). Attenuation by the rough ice boundaries was also included, although omitted here.

## FORMAL SOLUTION

### Source Free Case

Consider the n-layered interbedded liquid-solid half-space shown in Figure 1 in which the last layer of infinite thickness is liquid and has a higher sound velocity than liquid layers immediately above it; that is, a high-speed bottom. A point source of harmonic waves is located in one of the liquid layers. The velocity potential in the m-th layer, which is

assumed to be liquid, satisfies the wave equation

$$\nabla^2 \phi_m - \frac{1}{\alpha_m^2}\ddot{\phi}_m = 0.$$

In this layer

$$(p_{zz})_m = -p_m = \rho_m \frac{\partial \phi_m}{\partial t}, \quad \dot{w}_m = \frac{\partial \phi_m}{\partial z}, \quad \dot{u}_m = \frac{\partial \phi_m}{\partial r}.$$

Solutions for $\phi_m$, $\dot{w}_m$ and $(p_{zz})_m$ are given by

$$\phi_m(r,z,t,k) = \int_0^\infty \hat{\phi}_m(z)\, J_0(kr) e^{i\omega t} dk$$

$$\dot{w}_m(r,z,t,k) = \int_0^\infty \hat{\dot{w}}_m(z)\, J_0(kr) e^{i\omega t} dk$$

$$(p_{zz})_m(r,z,t,k) = \int_0^\infty (\hat{p}_{zz})_m(z)\, J_0(kr) e^{i\omega t} dk.$$

Separation of variables yields

$$\hat{\phi}_m(z) e^{i\omega t} = \left(A_m e^{i\sqrt{k_{\alpha_m}^2 - k^2}\, z} + A_m' e^{-i\sqrt{k_{\alpha_m}^2 - k^2}\, z}\right) e^{i\omega t}$$

$$\hat{\dot{w}}_m(z) e^{i\omega t} = \frac{\partial \hat{\phi}_m}{\partial z} e^{i\omega t} = \left(i\sqrt{k_{\alpha_m}^2 - k^2}\, A_m e^{i\sqrt{k_{\alpha_m}^2 - k^2}\, z}\right.$$

$$\left. - i\sqrt{k_{\alpha_m}^2 - k^2}\, A_m' e^{-i\sqrt{k_{\alpha_m}^2 - k^2}\, z}\right) e^{i\omega t}$$

and

$$(\hat{p}_{zz})(z)e^{i\omega t} = \rho_m \frac{\partial}{\partial t}\hat{\phi}_m e^{i\omega t} = i\omega\rho_m\left(A_m e^{i\sqrt{k_{d_m}^2-k^2}\,z} + A_m' e^{-i\sqrt{k_{d_m}^2-k^2}\,z}\right)e^{i\omega t}$$

where $A_m$ and $A_m'$ are constants.

Placing the origin at the (m-1)-st interface we get

$$\dot{\hat{w}}_{m-1} = i\sqrt{k_{d_m}^2-k^2}\,(A_m - A_m')$$

(1)

$$(\hat{p}_{zz})_{m-1} = i\omega\rho_m\,(A_m + A_m') \quad \text{for } z = 0$$

and

$$\dot{\hat{w}}_m = -i\sqrt{k_{d_m}^2-k^2}\,(A_m + A_m')\sin P_m + i\sqrt{k_{d_m}^2-k^2}\,(A_m - A_m')\cos P_m \qquad (2)$$

$$(\hat{p}_{zz})_m = i\omega\rho_m\,(A_m + A_m')\cos P_m - \omega\rho_m\,(A_m - A_m')\sin P_m$$

for $z = h_m$.

Substituting expressions for $[A_m - A_m']$ and $[A_m + A_m']$ from equations (1) in equations (2) yields

$$\dot{\hat{w}}_m = \dot{\hat{w}}_{m-1}\cos P_m - (\hat{p}_{zz})_{m-1}\frac{\sqrt{k_{d_m}^2 - k^2}}{i\omega\rho_m}\sin P_m$$

$$(\hat{p}_{zz})_m = -\dot{\hat{w}}_{m-1}\frac{\omega\rho_m}{i\sqrt{k_{d_m}^2 - k^2}}\sin P_m + (\hat{p}_{zz})_{m-1}\cos P_m$$

or in matrix notation

$$\begin{bmatrix} \dot{\hat{w}}_m \\ (\hat{p}_{zz})_m \end{bmatrix} = \begin{bmatrix} \cos P_m & \dfrac{i\sqrt{k_{d_m}^2 - k^2}}{\omega\rho_m}\sin P_m \\ \dfrac{i\omega\rho_m \sin P_m}{\sqrt{k_{d_m}^2 - k^2}} & \cos P_m \end{bmatrix} \cdot \begin{bmatrix} \dot{\hat{w}}_{m-1} \\ (\hat{p}_{zz})_{m-1} \end{bmatrix} =$$

$$\begin{bmatrix} (a_m)_{11} & (a_m)_{12} \\ (a_m)_{21} & (a_m)_{22} \end{bmatrix} \begin{bmatrix} \dot{\hat{w}}_{m-1} \\ (\hat{p}_{zz})_{m-1} \end{bmatrix}.$$

In general for an $n$-layered liquid half-space

$$\begin{bmatrix} \dot{\hat{w}}_{n-1} \\ (p_{zz})_{n-1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dot{\hat{w}}_0 \\ (\hat{p}_{zz})_0 \end{bmatrix}$$ or

$$\dot{\hat{w}}_{n-1} = A_{11}\dot{\hat{w}}_0 + A_{12}(\hat{p}_{zz})_0$$

$$(\hat{p}_{n-1})_{zz} = A_{21}\dot{\hat{w}}_0 + A_{22}(\hat{p}_{zz})_0$$ where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = a_{n-1}a_{n-2}\cdots a_1$$

$$= \begin{bmatrix} (a_{n-1})_{11} & (a_{n-1})_{12} \\ (a_{n-1})_{21} & (a_{n-1})_{22} \end{bmatrix} \cdots \begin{bmatrix} (a_1)_{11} & (a_1)_{12} \\ (a_1)_{21} & (a_1)_{22} \end{bmatrix}$$

since the vertical particle velocity and pressure are continuous across each interface. For the n-th layer

$$A_n - A_n' = \frac{\dot{\hat{w}}_{n-1}}{i\sqrt{k_{\alpha_n}^2 - k^2}} = \frac{A_{11}\dot{\hat{w}}_0}{i\sqrt{k_{\alpha_n}^2 - k^2}} + \frac{A_{12}(\hat{p}_{zz})_0}{i\sqrt{k_{\alpha_n}^2 - k^2}}$$

$$A_n + A_n' = \frac{(\hat{p}_{zz})_{n-1}}{i\omega\rho_n} = \frac{A_{21}\,\dot{\hat{w}}_0}{i\omega\rho_n} + \frac{A_{22}(\hat{p}_{zz})_0}{i\omega\rho_n}$$

or in matrix notation

$$\begin{bmatrix} A_n - A_n' \\ A_n + A_n' \end{bmatrix} = E^{-1} \begin{bmatrix} \dot{\hat{w}}_{n-1} \\ (\hat{p}_{zz})_{n-1} \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} \frac{1}{i\sqrt{k_{\alpha_n}^2 - k^2}} & 0 \\ 0 & \frac{1}{i\omega\rho_n} \end{bmatrix}. \tag{3}$$

Applying boundary conditions that the pressure vanishes at the surface and that no upgoing waves travel from infinity, $(p_{zz})_o = 0$, $A_n = 0$, the relation (3) reduces to

$$\begin{bmatrix} -A_n' \\ A_n' \end{bmatrix} = E^{-1} A \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix}.$$

Consider now a solid layer between two liquid layers or at the surface of the laminated halfspace. For this layer we may write (Thomson, 1950; Dorman, 1962)

$$\begin{bmatrix} (a_m)_{11} & (a_m)_{12} & (a_m)_{13} & (a_m)_{14} \\ (a_m)_{21} & (a_m)_{22} & (a_m)_{23} & (a_m)_{24} \\ (a_m)_{31} & (a_m)_{32} & (a_m)_{33} & (a_m)_{34} \\ (a_m)_{41} & (a_m)_{42} & (a_m)_{43} & (a_m)_{44} \end{bmatrix} \begin{bmatrix} \dot{\hat{u}}_{m-1} \\ \dot{\hat{w}}_{m-1} \\ (\hat{p}_{zz})_{m-1} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\hat{u}}_{m-1} \\ \dot{\hat{w}}_m \\ (\hat{p}_{zz})_m \\ 0 \end{bmatrix} \tag{4}$$

where the fourth element of the column vector is the tangential stress which is zero at a solid-liquid boundary. The matrix elements are given in the appendix and may be derived following Haskell (1953). Equation (4) yields the three equations.

$$\dot{\hat{u}}_{m-1} = -\frac{(a_m)_{42}}{(a_m)_{41}}\dot{\hat{w}}_{m-1} - \frac{(a_m)_{43}}{(a_m)_{41}}(\hat{p}_{zz})_{m-1} \tag{5}$$

$$\dot{\hat{w}}_m = (a_m)_{21}\dot{\hat{u}}_{m-1} + (a_m)_{22}\dot{\hat{w}}_{m-1} + (a_m)_{23}(\hat{p}_{zz})_{m-1} \tag{6}$$

$$(\hat{p}_{zz})_m = (a_m)_{31}\dot{\hat{u}}_{m-1} + (a_m)_{32}\dot{\hat{w}}_{m-1} + (a_m)_{33}(\hat{p}_{zz})_{m-1} . \tag{7}$$

Substituting $\dot{\hat{u}}_{m-1}$ from equation (5) in equations (6) and (7) yields

$$\dot{\hat{w}}_m = \left[(a_m)_{22} - \frac{(a_m)_{21}(a_m)_{42}}{(a_m)_{41}}\right]\dot{\hat{w}}_{m-1} +$$

$$\left[(a_m)_{23} - \frac{(a_m)_{21}(a_m)_{43}}{(a_m)_{41}}\right](\hat{p}_{zz})_{m-1}$$

$$(\hat{p}_{zz})_m = \left[(a_m)_{32} - \frac{(a_m)_{31}(a_m)_{42}}{(a_m)_{41}}\right]\dot{\hat{w}}_{m-1} +$$



$$\left[(a_m)_{33} - \frac{(a_m)_{31}(a_m)_{43}}{(a_m)_{41}}\right](\hat{p}_{zz})_{m-1}$$

or in matrix notation

$$\begin{bmatrix} \dot{\hat{w}}_m \\ (\hat{p}_{zz})_m \end{bmatrix} = \begin{bmatrix} (a_m)_{22} - \dfrac{(a_m)_{21}(a_m)_{42}}{(a_m)_{41}} & (a_m)_{23} - \dfrac{(a_m)_{21}(a_m)_{43}}{(a_m)_{41}} \\ (a_m)_{32} - \dfrac{(a_m)_{31}(a_m)_{42}}{(a_m)_{41}} & (a_m)_{33} - \dfrac{(a_m)_{31}(a_m)_{43}}{(a_m)_{41}} \end{bmatrix} \cdot$$

$$\begin{bmatrix} \dot{\hat{w}}_{m-1} \\ (\hat{p}_{zz})_{m-1} \end{bmatrix} = \begin{bmatrix} (b_m)_{11} & (b_m)_{12} \\ (b_m)_{21} & (b_m)_{22} \end{bmatrix} \begin{bmatrix} \dot{\hat{w}}_{m-1} \\ (\hat{p}_{zz})_{m-1} \end{bmatrix}$$

which is the same as the matrix relation for a liquid layer. In general, then, for interbedded solids and liquids

$$\begin{bmatrix} \dot{\hat{w}}_{n-1} \\ (\hat{p}_{zz})_{n-1} \end{bmatrix} = a_{n-1} \cdots b_{n-3}\, a_{n-4} \cdots b_0 \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix}$$

$$= A \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix}.$$

Point Source of Harmonic Waves in a Liquid Layer.

Divide the source layer into two layers as shown in Figure 1. At z=D the pressure is continuous. The vertical particle velocity is continuous everywhere in the plane defined by z =D except at the point source where the liquid above and below the source moves in opposite directions. This may be expressed by writing

$$\delta(\dot{\hat{w}})_s = 2k.$$

In matrix notation for the source layer

$$\begin{bmatrix} \dot{\hat{w}}_{s_2}(D) \\ (\hat{p}_{zz})_{s_2}(D) \end{bmatrix} = \begin{bmatrix} \dot{\hat{w}}_{s_1}(D) \\ (\hat{p}_{zz})_{s_1}(D) \end{bmatrix} + \begin{bmatrix} \delta(\dot{\hat{w}})_s \\ 0 \end{bmatrix}. \tag{8}$$

For the layers below z =D we have

$$\begin{bmatrix} \dot{\hat{w}}_{n-1} \\ (\hat{p}_{zz})_{n-1} \end{bmatrix} = A_{s_2} \begin{bmatrix} \dot{\hat{w}}_{s_2}(D) \\ (\hat{p}_{zz})_{s_2}(D) \end{bmatrix} \tag{9}$$

and for the layers above z =D

$$\begin{bmatrix} \dot{\hat{w}}_{s_1}(D) \\ (\hat{p}_{zz})_{s_1} \end{bmatrix} = A_{s_1} \begin{bmatrix} \dot{\hat{w}}_{0} \\ 0 \end{bmatrix} \tag{10}$$

where $A_{s_2} = a_{n-1}\, a_{n-2} \cdots a_{s_2}$ and $A_{s_1} = a_{s_1}\, a_{s_1 - 1} \cdots a_1$.

Returning now to the relation for the n-th liquid layer we write from equations (8), (9), and (10)

$$\begin{bmatrix} -A_n' \\ A_n' \end{bmatrix} = E^{-1} \begin{bmatrix} \hat{w}_{n-1} \\ (\hat{p}_{zz})_{n-1} \end{bmatrix} = E^{-1} A_{s_2} \begin{bmatrix} \dot{\hat{w}}_{s_2}(D) \\ (\hat{p}_{zz})_{s_2}(D) \end{bmatrix}$$

$$= E^{-1} A_{s_2} \left\{ \begin{bmatrix} \dot{\hat{w}}_{s_1}(D) \\ (\hat{p}_{zz})_{s_1}(D) \end{bmatrix} + \begin{bmatrix} \delta(\dot{\hat{w}})_s \\ 0 \end{bmatrix} \right\}$$

$$= E^{-1} A_{s_2} \left\{ A_{s_1} \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \delta(\dot{\hat{w}})_s \\ 0 \end{bmatrix} \right\}$$

$$= E^{-1} \left\{ A_{s_2} A_{s_1} \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix} + A_{s_2} \begin{bmatrix} \delta(\dot{\hat{w}})_s \\ 0 \end{bmatrix} \right\}. \tag{11}$$

In terms of the inverse matrix $A_{s_1}^{-1}$ of $A_{s_1}$ (11) may be written

$$\begin{bmatrix} -A_n' \\ A_n' \end{bmatrix} = E^{-1} A \left\{ \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix} + A_{s_1}^{-1} \begin{bmatrix} \delta(\dot{\hat{w}})_s \\ 0 \end{bmatrix} \right\} \tag{12}$$

where $A = A_{S_2} A_{S_1} = a_{n-1} a_{n-2} \cdots b_{n-10} \cdots a_0$

Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix} + A_{S_1}^{-1} \begin{bmatrix} \delta(\dot{\hat{w}})_S \\ 0 \end{bmatrix}.$$

Harkrider (1964) has shown that for layered solids the inverse of the product matrix has the same form as the inverse of the layer matrices. Employing an extension of this theorem to inter-bedded liquids and solids we get

$$A_{S_1}^{-1} = \begin{bmatrix} (A_{S_1})_{22} & -(A_{S_1})_{12} \\ -(A_{S_1})_{21} & (A_{S_1})_{11} \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} (A_{S_1})_{22} & -(A_{S_1})_{12} \\ -(A_{S_1})_{21} & (A_{S_1})_{11} \end{bmatrix} \begin{bmatrix} \delta(\dot{\hat{w}})_S \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\hat{w}}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} (A_{S_1})_{22}\,\delta(\dot{\hat{w}})_S \\ -(A_{S_1})_{21}\,\delta(\dot{\hat{w}})_S \end{bmatrix}$$ or

$$X = \dot{\hat{\omega}}_0 + (A_{S_1})_{22}\,\delta(\dot{\hat{\omega}})_S$$

$$Y = -(A_{S_1})_{21}\,\delta(\dot{\hat{\omega}})_S \,. \tag{13}$$

We now define $J = E^{-1}A = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$.

Hence (12) may be written as

$$\begin{bmatrix} -A_n' \\ A_n' \end{bmatrix} = J\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} J_{11}X + J_{12}Y \\ J_{21}X + J_{22}Y \end{bmatrix}$$ or

$$-A_n' = J_{11}X + J_{12}Y$$

$$A_n' = J_{21}X + J_{22}Y\,. \tag{14}$$

Adding equations (14) we get

$0 = (J_{11} + J_{21})\,X + (J_{12} + J_{22})\,Y$ and if we

let $T = J_{11} + J_{21}$ and $V = J_{12} + J_{22}$ and solve for X we have

$$X = -\frac{VY}{T}\,. \tag{15}$$

Carrying out the matrix multiplication $J = E^{-1} A$, expressions for T and V are given by

$$T = \frac{A_{11}}{i\sqrt{k_{d_n}^2 - k^2}} + \frac{A_{21}}{i\omega\rho_n}$$
$$V = \frac{A_{12}}{i\sqrt{k_{d_n}^2 - k^2}} + \frac{A_{22}}{i\omega\rho_n} . \qquad (16)$$

From the first equation of (13)

$$\dot{\hat{w}}_0 = X - (A_{s_1})_{22}\,\delta(\dot{\hat{w}})_s \quad \text{and}$$

from equation (15)

$$\dot{\hat{w}}_0 = -\frac{VY}{T} - (A_{s_1})_{22}\,\delta(\dot{\hat{w}})_s .$$

Hence, from the second equation of (13) and equations (16)

$$\dot{\hat{w}}_0 = \frac{\left(\dfrac{A_{12}}{i\sqrt{k_{d_n}^2 - k^2}} + \dfrac{A_{22}}{i\omega\rho_n}\right)(A_{s_1})_{21}\,\delta(\dot{\hat{w}})_s}{\dfrac{A_{11}}{i\sqrt{k_{d_n}^2 - k^2}} + \dfrac{A_{21}}{i\omega\rho_n}}$$

$$- (A_{s_1})_{22}\,\delta(\dot{\hat{w}})_s .$$

Therefore the integral solution for the surface verticle particle velocity is $\dot{w}_0(r,z,k,t) =$

$$\int_0^\infty e^{i\omega t}\left\{\frac{\left[\frac{A_{12}}{i\sqrt{k_{\alpha_n}^2-k^2}}+\frac{A_{22}}{i\omega\rho_n}\right](A_{S_1})_{21}-\left[\frac{A_{11}}{i\sqrt{k_{\alpha_n}^2-k^2}}+\frac{A_{21}}{i\omega\rho_n}\right](A_{S_1})_{22}}{\frac{A_{11}}{i\sqrt{k_{\alpha_n}^2-k^2}}+\frac{A_{21}}{i\omega\rho_n}}\right\}\cdot 2J_0(kr)k\,dk$$

and from the formula

$(\hat{p}_{zz})_{D_1} = (A_{D_1})_{21}\hat{\dot{w}}_0 = -p_{D_1}$ the integral solution for pressure at depth $D_1$ is $p_{D_1}(r,z,k,t) =$

$$-\int_0^\infty e^{i\omega t}\left\{\frac{\left[\frac{A_{12}}{i\sqrt{k_{\alpha_n}^2-k^2}}+\frac{A_{22}}{i\omega\rho_n}\right](A_{S_1})_{21}}{\frac{A_{11}}{i\sqrt{k_{\alpha_n}^2-k^2}}+\frac{A_{21}}{i\omega\rho_n}} - \frac{\left[\frac{A_{11}}{i\sqrt{k_{\alpha_n}^2-k^2}}+\frac{A_{21}}{i\omega\rho_n}\right](A_{S_1})_{22}}{\frac{A_{11}}{i\sqrt{k_{\alpha_n}^2-k^2}}+\frac{A_{21}}{i\omega\rho_n}}\right\}(A_{D_1})_{21}\,2J_0(kr)k\,dk$$

where $A_{D_1} = a_{D_1}a_{D_1-1}\cdots a_0$.

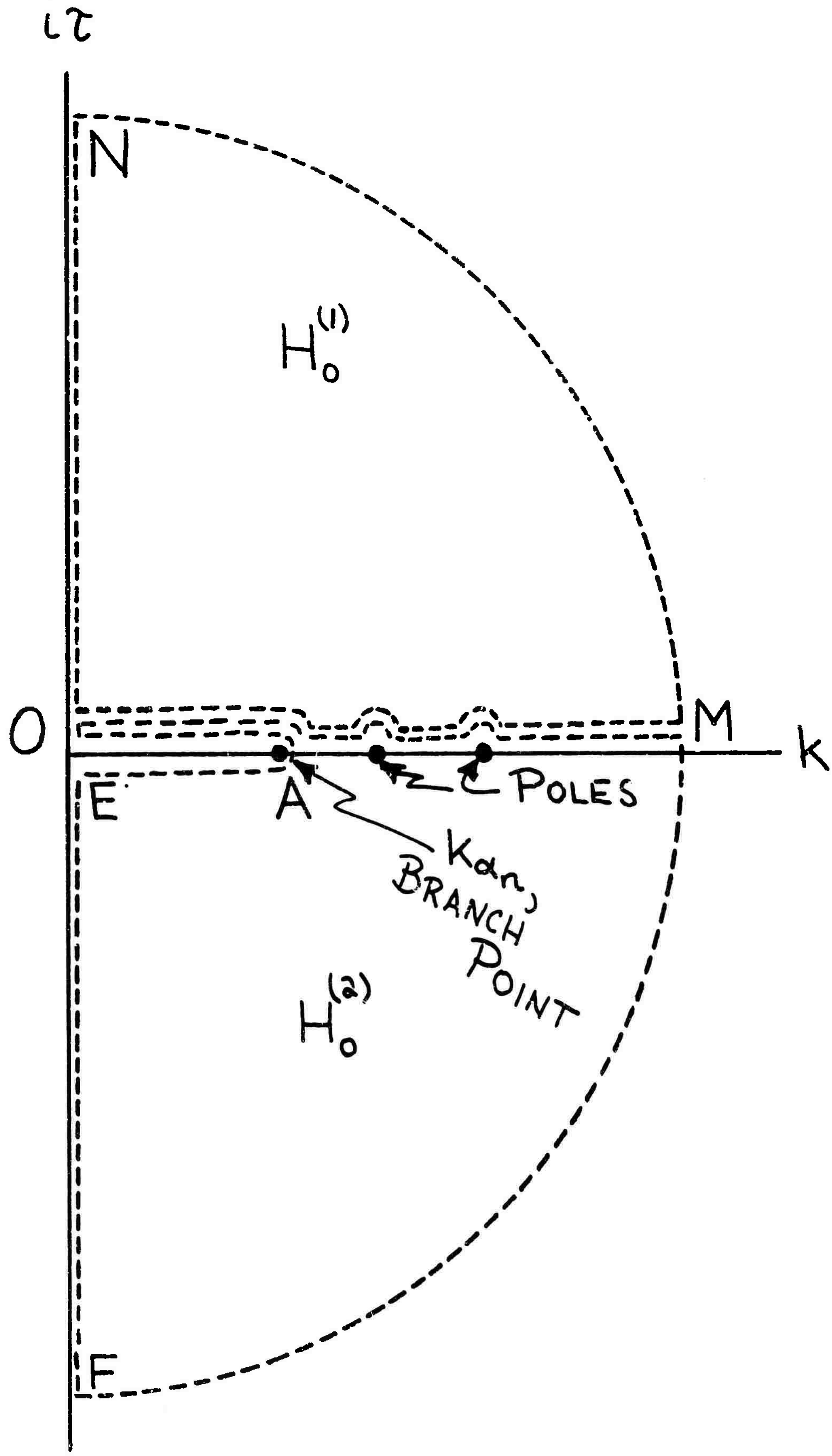


Fig. 2

Evaluation of the Integral Solutions.

The integral solutions have singularities corresponding to zeros of $\dfrac{A_{11}}{i\sqrt{k_{\alpha_n}^2-k^2}} + \dfrac{A_{21}}{i\omega\rho_n}$.

These are simple poles for complex wave number and, therefore, the integral solutions are conveniently evaluated by contour integration in the complex $\zeta = k + i\tau$ plane. The contours are shown in Figure 2. First the transformation

$$2J_0(\zeta r) = H_0^{(1)}(\zeta r) + H_0^{(2)}(\zeta r)$$ is made.

In addition to simple poles at

$$\frac{A_{11}}{i\sqrt{k_{\alpha_n}^2-\zeta^2}} + \frac{A_{21}}{i\omega\rho_n}$$

branch points occur at

$$\sqrt{k_{\alpha_m}^2-k^2} = 0$$

for the liquid layers and

$$\sqrt{k_{\alpha_m}^2-k^2} = 0 \text{ and } \sqrt{k_{\beta_m}^2-k^2} = 0$$

for each solid layer. But since the integrands are even functions of $\sqrt{k_{\alpha_m}^2-k^2}$ and $\sqrt{k_{\beta_m}^2-k^2}$

branch line integrals corresponding to branch cuts made to these branch points cancel and only the integral corresponding to the branch cut for $\sqrt{k_{\alpha_n}^2-k^2} = 0$ must be considered.

The Reimann surface has two sheets. To satisfy the vanishing of $\dot{w}_0$ as $\zeta$ approaches infinity, we must remain on the sheet of the Riemann surface where the real part of $i\beta_n$ is greater than zero. For the contours shown in Figure 2 complex poles of

$$\frac{A_{11}}{i\sqrt{k_{a_n}^2-k^2}}+\frac{A_{21}}{i\omega\rho_n}=0$$

have been displaced to an unused Reimann sheet and all poles are on the real k-axis. See Ewing, Jardetzky and Press (1957 , pages 135 to 137) for the proof for a two-layer liquid half space. Along the branch cut $i\sqrt{k_{d_n}^2-\zeta^2}$ is pure imaginary and along the real axis from $k_{d_n}$ to infinity $i\left(-i\sqrt{\zeta^2-k_{d_n}^2}\right)$ is real.

In the first quadrant $\mathrm{Im}\left[i\sqrt{k_{d_n}^2-\zeta^2}\right]$ is positive and in the fourth quadrant negative. The integral for $\dot{w}_0$ (omitting the $e^{i\omega t}$ term) is now transformed to

$$\int_0^M I_1\,d\zeta+\int_{C(M,N)} I_1\,d\zeta+\int_N^0 I_1\,d\zeta=0 \qquad (17)$$

and

$$\int_E^F I_2\,d\zeta+\int_{C(F,M)} I_2\,d\zeta+\int_M^0 I_2\,d\zeta+\int_{OAE} I_2\,d\zeta= 2\pi i\sum \mathrm{RES}(I_2) \qquad (18)$$

where

$$I_1 = \left\{ \frac{\left[\frac{A_{12}}{i\sqrt{k_{d_n}^2-\xi^2}} + \frac{A_{22}}{i\omega\rho_n}\right](A_{s_1})_{21} - \left[\frac{A_{11}}{i\sqrt{k_{d_n}^2-\xi^2}} + \frac{A_{21}}{i\omega\rho_n}\right](A_{s_1})_{22}}{\frac{A_{11}}{i\sqrt{k_{d_n}^2-\xi^2}} + \frac{A_{21}}{i\omega\rho_n}} \right\} \cdot H_0^{(1)}(\xi r)\xi$$

$$I_2 = \left\{ \frac{\left[\frac{A_{12}}{i\sqrt{k_{d_n}^2-\xi^2}} + \frac{A_{22}}{i\omega\rho_n}\right](A_{s_1})_{21} - \left[\frac{A_{11}}{i\sqrt{k_{d_n}^2-\xi^2}} + \frac{A_{21}}{i\omega\rho_n}\right](A_{s_1})_{22}}{\frac{A_{11}}{i\sqrt{k_{d_n}^2-\xi^2}} + \frac{A_{21}}{i\omega\rho_n}} \right\} \cdot H_0^{(2)}(\xi r)\xi .$$

Since $H_0^{(1)}(\xi r)$ approaches zero as $\xi$ approaches infinity in the first quadrant, (17) is

$$\int_0^\infty I_1 d\xi + \int_{i\infty}^0 I_1 d\xi = 0 \quad \text{or}$$

$$\int_0^\infty I_1 d\xi = -\int_{i\infty}^0 I_1 d\xi . \tag{19}$$

Likewise, since $H_0^{(2)}(\xi r)$ approaches zero as $\xi$ approaches infinity in the fourth quadrant (18) is

$$\int_0^{-i\infty} I_2 d\xi + \int_\infty^0 I_2 d\xi + \int_{OAE} I_2 d\xi =$$

$$2\pi i \sum \mathrm{Res}(I_2) \quad \text{or}$$

$$\int_0^\infty I_2 d\zeta = \int_0^{-i\infty} I_2 d\zeta + \int_{AOE} I_2 d\zeta = - 2\pi i \sum \mathrm{Res}(I_2) . \qquad (20)$$

Using the identity $H_0^{(1)}(\zeta r) = -H_0^{(2)}(-\zeta r)$ (19) may be written

as $$\int_0^\infty I_1 dk = \int_0^\infty F(i\sqrt{k_{d_n}^2+\tau^2})\, H_0^{(2)}(-i\tau r)\,\tau d\tau$$

and (20) may be written as

$$\int_0^\infty I_2 dk = -\int_0^\infty F(-i\sqrt{k_{d_n}^2+\tau^2})\, H_0^{(2)}(-i\tau r)\,\tau d\tau$$

$$+ \int_0^{k_{d_n}} \left[F(i\sqrt{k_{d_n}^2-k^2}) - F(-i\sqrt{k_{d_n}^2-k^2})\right] H_0^{(2)}(kr)\,k dk$$

$$- 2\pi i \sum \mathrm{Res}(I_2) .$$

Hence $$\dot{w}_0 = \int_0^\infty I_1 dk + \int_0^\infty I_2 dk =$$

$$\int_0^\infty \left[F(i\sqrt{k_{d_n}^2+\tau^2}) - F(-i\sqrt{k_{d_n}^2+\tau^2})\right] H_0^{(2)}(-i\tau r)\,\tau d\tau$$

$$+ \int_0^{k_{d_n}} \left[F(i\sqrt{k_{d_n}^2-k^2}) - F(-i\sqrt{k_{d_n}^2-k^2})\right] H_0^{(2)}(kr)\,k dk$$

$$- 2\pi i \sum \mathrm{Res}(I_2) .$$

In the first integral

$$F(i\sqrt{k_{\alpha_n}^2+\tau^2}) - F(-i\sqrt{k_{\alpha_n}^2+\tau^2}) = \frac{2(A_{11}A_{22}-A_{12}A_{21})(A_{s_1})_{21}}{\omega\rho_n\sqrt{k_{\alpha_n}^2+\tau^2}\left(\frac{A_{11}^2}{k_{\alpha_n}^2+\tau^2} - \frac{A_{21}^2}{\omega^2\rho_n^2}\right)}$$

but $A_{11}\,A_{22} - A_{12}\,A_{21} = 1$

and therefore

$$F(i\sqrt{k_{\alpha_n}^2+\tau^2}) - F(-i\sqrt{k_{\alpha_n}^2+\tau^2}) = \frac{2\omega\rho_n\sqrt{k_{\alpha_n}^2+\tau^2}(A_{s_1})_{21}}{\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2+\tau^2)A_{21}^2}$$

Likewise, in the second integral

$$F(i\sqrt{k_{\alpha_n}^2-k^2}) - F(-i\sqrt{k_{\alpha_n}^2-k^2}) = \frac{2\omega\rho_n\sqrt{k_{\alpha_n}^2-k^2}(A_s)_{21}}{\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2-k^2)A_{21}^2}$$

Therefore, the integral solution reduces to

$$\dot{w}_0 = \int_0^{\infty}\frac{2\omega\rho_n\sqrt{k_{\alpha_n}^2+\tau^2}(A_{s_1})_{21}H_0^{(2)}(-i\tau r)\tau\,d\tau}{\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2+\tau^2)A_{21}^2}$$

$$+\int_0^{k_{\alpha_n}}\frac{2\omega\rho_n\sqrt{k_{\alpha_n}^2-k^2}(A_{s_1})_{21}H_0^{(2)}(kr)k\,dk}{\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2-k^2)A_{21}^2}$$

$$-2\pi i\sum \mathrm{RES}(I_2)\,.$$

From the formulas $H_0^{(2)}(-i\tau r) = \frac{2i}{\pi} K_0(\tau r)$ and

$$H_0^{(2)}(kr) = J_0(kr) - iY_0(kr)$$

and multiplying $\dot{w}_0$ by $\frac{P_0}{i\omega\rho_s}$ for a constant pressure source of pressure amplitude $P_0$ at unit distance from the source

$$\dot{w}_0 = \int_0^\infty \frac{4P_0\omega\rho_n\sqrt{k_{\alpha_n}^2+\tau^2}\,(A_{s_1})_{21} K_0(\tau r)\tau d\tau}{\pi\omega\rho_s[\omega^2\rho_n^2 A_{11}^2 - (k_{\alpha_n}^2+\tau^2)A_{21}^2]}$$

$$+ \int_0^{k_{\alpha_n}} \frac{2P_0\omega\rho_n\sqrt{k_{\alpha_n}^2-k^2}\,(A_{s_1})_{21}[J_0(kr)-iY_0(kr)]kdk}{i\omega\rho_s[\omega^2\rho_n^2 A_{11}^2 - (k_{\alpha_n}^2-k^2)A_{21}^2]}$$

$$- 2\pi i \sum \mathrm{RES}\left(\frac{P_0 I_2}{i\omega\rho_s}\right).$$

For the normal mode contribution, $-2\pi i \sum \mathrm{RES}\left(\frac{P_0 I_2}{i\omega\rho_s}\right)$, it is convenient to rewrite the integral for $\dot{w}_0$ in the form

$$\dot{w}_0 = \int_0^\infty \left\{ \frac{\left[\frac{c\bar{A}_{12}}{ikr_{\alpha_n}} + \frac{\bar{A}_{22}}{i\omega\rho_n}\right](A_{s_1})_{21} - \left[\frac{c\bar{A}_{11}}{ikr_{\alpha_n}} + \frac{\bar{A}_{21}}{i\omega\rho_n}\right](A_{s_1})_{22}}{\left[\frac{c\bar{A}_{11}}{ikr_{\alpha_n}} + \frac{\bar{A}_{21}}{i\omega\rho_n}\right]} \right\} 2J_0(kr)kdk.$$

where the layer matrices $\overline{a}_m$ are defined in the Appendix. There are, as before, simple poles of $I_2$ along the real axis at

$$\frac{c\bar{A}_{11}}{ikr_{\alpha_n}} + \frac{\bar{A}_{21}}{i\omega\rho_n} = 0 \quad \text{for } \omega_\ell, k_\ell, \ell = 1, 2, \ldots .$$

This is the period equation for the laminated half space. From the period equation we write

$$\left(\frac{i\omega\rho_n c}{ikr_{\alpha_n}}\right)_{\substack{\omega=\omega_\ell \\ k=k_\ell}} = \left(-\frac{\bar{A}_{21}}{\bar{A}_{11}}\right)_{\substack{\omega=\omega_\ell \\ k=k_\ell}}$$

and hence

$$-2\pi i \sum \operatorname{Res}\left(\frac{P_0 I_2}{i\omega\rho_s}\right) =$$

$$-2\pi i \sum_\ell \left\{ \frac{P_0\left[\bar{A}_{22} - \frac{\bar{A}_{21}\bar{A}_{12}}{\bar{A}_{11}}\right](\bar{A}_{s_1})_{21} H_0^{(2)}(kr)k}{i\omega\rho_s \frac{\partial}{\partial k}\left[\frac{c\bar{A}_{11}}{ikr_{\alpha_n}} + \frac{\bar{A}_{21}}{i\omega\rho_n}\right]} \left[\frac{1}{i\omega\rho_n}\right] \right\}_{\substack{\omega=\omega_\ell \\ k=k_\ell}} .$$

But again $\bar{A}_{11}\bar{A}_{22} - \bar{A}_{21}\bar{A}_{12} = 1$, and therefore

$$-2\pi i \sum \operatorname{Res}\left(\frac{P_0 I_2}{i\omega\rho_s}\right) =$$

$$2\pi \sum_\ell \left\{ \frac{P_0\omega(\bar{A}_{s_1})_{21}\left[J_0(kr) - iY_0(kr)\right]}{c^3\rho_s\rho_n\bar{A}_{11}\frac{\partial}{\partial c}\left[\frac{c^2\bar{A}_{11}}{r_{\alpha_n}} + \frac{\bar{A}_{21}}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_\ell \\ c=c_\ell}}$$

In summary, the expressions for vertical particle velocity at the surface, vertical particle displacement at the surface, and pressure in a liquid layer are:

$$\dot{w}_0 = \int_0^\infty \frac{4P_0\rho_n\sqrt{k_{\alpha_n}^2+\zeta^2}\,(A_{s_1})_{21}\,K_0(\zeta r)\,\zeta\,d\zeta}{\pi\rho_s\left[\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2+\zeta^2)A_{21}^2\right]} +$$

$$\int_0^{k_{\alpha_n}} \frac{2P_0\rho_n\sqrt{k_{\alpha_n}^2-k^2}\,(A_{s_1})_{21}\left[J_0(kr)-iY_0(kr)\right]k\,dk}{i\rho_s\left[\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2-k^2)A_{21}^2\right]}$$

$$+\,2\pi\sum_{l}\left\{\frac{P_0\omega(\bar{A}_{s_1})_{21}\left[J_0(kr)-iY_0(kr)\right]}{c^3\rho_s\rho_n\,\bar{A}_{11}\frac{\partial}{\partial c}\left[\frac{c^2\bar{A}_{11}}{r_{\alpha_n}}+\frac{\bar{A}_{21}}{\rho_n}\right]}\right\}_{\substack{\omega=\omega_l\\ c=c_l}}$$

$$w_0 = \frac{\dot{w}_0}{i\omega} = \int_0^\infty \frac{4P_0\rho_n\sqrt{k_{\alpha_n}^2+\zeta^2}\,(A_{s_1})_{21}\,K_0(\zeta r)\,\zeta\,d\zeta}{i\pi\rho_s\omega\left[\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2+\zeta^2)A_{21}^2\right]}$$

$$-\int_0^{k_{\alpha_n}} \frac{2P_0\rho_n\sqrt{k_{\alpha_n}^2-k^2}\,(A_{s_1})_{21}\left[J_0(kr)-iY_0(kr)\right]k\,dk}{\omega\rho_s\left[\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2-k^2)A_{21}^2\right]}$$

$$+2\pi\sum_{\ell}\left\{\frac{P_0\omega(\bar{A}_{s_1})_{21}\left[J_0(kr)-iY_0(kr)\right]}{i\omega c^3\rho_s\rho_n\bar{A}_{11}\frac{\partial}{\partial c}\left[\frac{c^2\bar{A}_{11}}{r_{\alpha_n}}+\frac{\bar{A}_{21}}{\rho_n}\right]}\right\}_{\substack{\omega=\omega_\ell\\ c=c_\ell}}$$

$$p_{D_1}=-\int_0^\infty\frac{4P_0\rho_n\sqrt{k_{\alpha_n}^2+\tau^2}\,(A_{s_1})_{21}(A_{D_1})_{21}K_0(\tau r)\tau d\tau}{\pi\rho_s\left[\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2+\tau^2)A_{21}^2\right]}$$

$$+i\int_0^{k_{\alpha_n}}\frac{2P_0\rho_n\sqrt{k_{\alpha_n}^2-k^2}\,(A_{s_1})_{21}(A_{D_1})_{21}\left[J_0(kr)-iY_0(kr)\right]kdk}{\rho_s\left[\omega^2\rho_n^2A_{11}^2-(k_{\alpha_n}^2-k^2)A_{21}^2\right]}$$

$$-2\pi\sum_{\ell}\left\{\frac{P_0\omega(\bar{A}_{s_1})_{21}(\bar{A}_{D_1})_{21}\left[J_0(kr)-iY_0(kr)\right]}{c^4\rho_s\rho_n\bar{A}_{11}\frac{\partial}{\partial c}\left[\frac{c^2\bar{A}_{11}}{r_{\alpha_n}}+\frac{\bar{A}_{21}}{\rho_n}\right]}\right\}_{\substack{\omega=\omega_\ell\\ c=c_\ell}}$$

where for the normal mode terms for $p_{D_1}$ the relation $p_{D_1} = -(\bar{A}_{D_1})_{21}\left(\frac{i\omega_0}{c}\right)$ has been used.

For programming it is convenient to write:

$$\dot{w}_0 = -W_1 + W_2 - iW_3 + iW_4 - iW_5$$

and for the absolute value of vertical particle velocity, a quantity conveniently measured in transmission experiments,

$$|\dot{w}_0| = \sqrt{(-W_1 + W_2)^2 + (-W_3 + W_4 - W_5)^2}$$

where

$$W_1 = \int_0^{k_{\alpha n}} \frac{2P_0 \rho_n \sqrt{k_{\alpha n}^2 - k^2}\, Im\left[-(A_{s_1})_{21}\right] J_0(kr)\, k\, dk}{\rho_s \left\{ \omega^2 \rho_n^2 A_{11}^2 + (k_{\alpha n}^2 - k^2)\left[Im(-A_{21})\right]^2 \right\}}$$

$$W_2 = 2\pi \sum_{\ell} \left\{ \frac{P_0 \omega\, Im\left[-(\bar{A}_{s_1})_{21}\right] J_0(kr)}{c^3 \rho_s \rho_n \bar{A}_{11} \frac{\partial}{\partial c}\left[ \frac{c^2 \bar{A}_{11}}{Im(r_{\alpha n})} + \frac{Im(-\bar{A}_{21})}{\rho_n} \right]} \right\}_{\substack{\omega = \omega_\ell \\ c = c_\ell}}$$

$$W_3 = \int_0^{\infty} \frac{4P_0 \rho_n \sqrt{k_{\alpha n}^2 + \tau^2}\, Im\left[-(A_{s_1})_{21}\right] K_0(\tau r)\, \tau\, d\tau}{\pi \rho_s \left\{ \omega^2 \rho_n^2 A_{11}^2 + (k_{\alpha n}^2 + \tau^2)\left[Im(-A_{21})\right]^2 \right\}}$$

$$W_4 = \int_0^{k_{\alpha n}} \frac{2P_0 \rho_n \sqrt{k_{\alpha n}^2 - k^2}\, Im\left[-(A_{s_1})_{21}\right] Y_0(kr)\, k\, dk}{\rho_s \left\{ \omega^2 \rho_n^2 A_{11}^2 + (k_{\alpha n}^2 - k^2)\left[Im(-A_{21})\right]^2 \right\}}$$

$$W_5 = 2\pi \sum \left\{ \frac{P_o \omega \, Im[-(\bar{A}_{s_1})_{21}] \, Y_0(kr)}{c^3 \rho_s \rho_n \bar{A}_{11} \frac{\partial}{\partial c}\left[\frac{c^2 \bar{A}_{11}}{Im(r_{\alpha_n})} + \frac{Im(-\bar{A}_{21})}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_\ell \\ c=c_\ell}}$$

Likewise for vertical particle displacement

$$w_o = -W_6 + W_7 - W_8 + iW_9 - iW_{10}$$

$$|w_o| = \sqrt{(-W_6 + W_7 - W_8)^2 + (W_9 - W_{10})^2}$$

$$W_6 = \int_0^\infty \frac{4 P_o \rho_n \sqrt{k_{\alpha_n}^2 + \tau^2} \, Im[-(A_{s_1})_{21}] \, K_0(\tau r) \tau d\tau}{\pi \rho_s \omega \left\{ \omega^2 \rho_n^2 A_{11}^2 + (k_{\alpha_n}^2 + \tau^2)[Im(-A_{21})]^2 \right\}}$$

$$W_7 = \int_0^{k_{\alpha_n}} \frac{2 P_o \rho_n \sqrt{k_{\alpha_n}^2 - k^2} \, Im[-(A_{s_1})_{21}] \, Y_0(kr) k dk}{\omega \rho_s \left\{ \omega^2 \rho_n^2 A_{11}^2 + (k_{\alpha_n}^2 - k^2)[Im(-A_{21})]^2 \right\}}$$

$$W_8 = 2\pi \sum_\ell \left\{ \frac{P_o \, Im[(-\bar{A}_{s_1})_{21}] \, Y_0(kr)}{c^3 \rho_s \rho_n \bar{A}_{11} \frac{\partial}{\partial c}\left[\frac{c^2 \bar{A}_{11}}{Im(r_{\alpha_n})} + \frac{Im(-\bar{A}_{21})}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_\ell \\ c=c_\ell}}$$

$$W_9 = \int_0^{K_{\alpha_n}} \frac{2P_0\rho_n\sqrt{K_{\alpha_n}^2-k^2}\,\mathrm{Im}\left[-(A_{S_1})_{21}\right] J_0(kr)\,k\,dk}{\omega\rho_s\left\{\omega^2\rho_n^2A_{11}^2+(K_{\alpha_n}^2-k^2)\left[\mathrm{Im}(-A_{21})\right]^2\right\}}$$

$$W_{10} = 2\pi\sum_{\ell}\left\{\frac{P_0\,\mathrm{Im}\left[-(\bar{A}_{S_1})_{21}\right] J_0(kr)}{c^3\rho_s\rho_n\bar{A}_{11}\frac{\partial}{\partial c}\left[\frac{c^2\bar{A}_{11}}{\mathrm{Im}(r_{\alpha_n})}+\frac{\mathrm{Im}(-\bar{A}_{21})}{\rho_n}\right]}\right\}_{\substack{\omega=\omega_\ell\\ c=c_\ell}}$$

and for pressure

$$p_{D_1} = W_{11} - W_{12} + W_{13} - iW_{14} + iW_{15}$$

$$|p_{D_1}| = \sqrt{(W_{11}-W_{12}+W_{13})^2+(-W_{14}+W_{15})^2}$$

$$W_{11} =$$

$$\int_0^{\infty}\frac{4P_0\rho_n\sqrt{K_{\alpha_n}^2+\tau^2}\,\mathrm{Im}\left[-(A_{S_1})_{21}\right]\mathrm{Im}\left[-(A_D)_{21}\right]K_0(\tau r)\,\tau\,d\tau}{\pi\rho_s\left\{\omega^2\rho_n^2A_{11}^2+(K_{\alpha_n}^2+\tau^2)\left[\mathrm{Im}(-A_{21})\right]^2\right\}}$$

$$W_{12} =$$

$$\int_0^{k_{\alpha_n}} \frac{2 P_0 \rho_n \sqrt{k_{\alpha_n}^2 - k^2}\, Im\left[-(A_{S_1})_{21}\right] Im\left[-(A_D)_{21}\right] Y_0(kr)\, k\, dk}{\rho_s \left\{ \omega^2 \rho_n^2 A_{11}^2 + (k_{\alpha_n}^2 - k^2)\left[Im(-A_{21})\right]^2 \right\}}$$

$$W_{13} =$$

$$2\pi \sum_{\ell} \left\{ \frac{P_0 \omega\, Im\left[-(\bar{A}_{S_1})_{21}\right] Im\left[-(\bar{A}_{D_1})_{21}\right] Y_0(kr)}{c^4 \rho_s \rho_n \bar{A}_{11} \frac{\partial}{\partial c}\left[ \frac{c^2 \bar{A}_{11}}{Im(r_{\alpha_n})} + \frac{Im(-\bar{A}_{21})}{\rho_n} \right]} \right\}_{\substack{\omega=\omega_\ell \\ c=c_\ell}}$$

$$W_{14} =$$

$$\int_0^{k_{\alpha_n}} \frac{2 P_0 \rho_n \sqrt{k_{\alpha_n}^2 - k^2}\, Im\left[-(A_{S_1})_{21}\right] Im\left[-(A_{D_1})_{21}\right] J_0(kr)\, k\, dk}{\rho_s \left\{ \omega^2 \rho_n^2 A_{11}^2 + (k_{\alpha_n}^2 - k^2)\left[Im(-A_{21})\right]^2 \right\}}$$

$$W_{15} =$$

$$2\pi \sum_{\ell} \left\{ \frac{P_0 \omega\, Im\left[-(\bar{A}_{S_1})_{21}\right] Im\left[-(\bar{A}_{D_1})_{21}\right] J_0(kr)}{c^4 \rho_s \rho_n \bar{A}_{11} \frac{\partial}{\partial c}\left[ \frac{c^2 \bar{A}_{11}}{Im(r_{\alpha_n})} + \frac{Im(-\bar{A}_{21})}{\rho_n} \right]} \right\}_{\substack{\omega=\omega_\ell \\ c=c_\ell}}$$

The layer matrices for the integral along the real axis are programmed as

$$\begin{bmatrix} (a_m)_{11} & -Im[(a_m)_{12}] & (a_m)_{13} & -Im[(a_m)_{14}] \\ Im[(a_m)_{21}] & (a_m)_{22} & Im[(a_m)_{23}] & (a_m)_{24} \\ (a_m)_{31} & -Im[(a_m)_{32}] & (a_m)_{33} & -Im[(a_m)_{34}] \\ Im[(a_m)_{41}] & (a_m)_{42} & Im[(a_m)_{43}] & (a_m)_{44} \end{bmatrix}$$

for solid layers and

$$\begin{bmatrix} (a_m)_{11} & Im[(a_m)_{12}] \\ -Im[(a_m)_{21}] & (a_m)_{22} \end{bmatrix}$$

for liquid layers. For the integral along the imaginary axis the matrices are

$$\begin{bmatrix} (a_m)_{11} & (a_m)_{12} & -\mathrm{Im}\left[(a_m)_{13}\right] & -\mathrm{Im}\left[(a_m)_{14}\right] \\ (a_m)_{21} & (a_m)_{22} & -\mathrm{Im}\left[(a_m)_{23}\right] & -\mathrm{Im}\left[(a_m)_{24}\right] \\ \mathrm{Im}\left[(a_m)_{31}\right] & \mathrm{Im}\left[(a_m)_{32}\right] & (a_m)_{33} & (a_m)_{34} \\ \mathrm{Im}\left[(a_m)_{41}\right] & \mathrm{Im}\left[(a_m)_{42}\right] & (a_m)_{43} & (a_m)_{44} \end{bmatrix}$$

for solid layers and

$$\begin{bmatrix} (a_m)_{11} & \mathrm{Im}\left[(a_m)_{12}\right] \\ -\mathrm{Im}\left[(a_m)_{21}\right] & (a_m)_{22} \end{bmatrix}$$

for liquid layers. Likewise, for the normal mode terms the layer matrices are programmed as

$$\begin{bmatrix} (\bar{a}_m)_{11} & -\mathrm{Im}\left[(\bar{a}_m)_{12}\right] & (\bar{a}_m)_{13} & -\mathrm{Im}\left[(\bar{a}_m)_{14}\right] \\ \mathrm{Im}\left[(\bar{a}_m)_{21}\right] & (\bar{a}_m)_{22} & \mathrm{Im}\left[(\bar{a}_m)_{23}\right] & (\bar{a}_m)_{24} \\ (\bar{a}_m)_{31} & -\mathrm{Im}\left[(\bar{a}_m)_{32}\right] & (\bar{a}_m)_{33} & -\mathrm{Im}\left[(\bar{a}_m)_{34}\right] \\ \mathrm{Im}\left[(\bar{a}_m)_{41}\right] & (\bar{a}_m)_{42} & \mathrm{Im}\left[(\bar{a}_m)_{43}\right] & (\bar{a}_m)_{44} \end{bmatrix}$$

for solid layers and

$$\begin{bmatrix} (\bar{a}_m)_{11} & \mathrm{Im}\left[(\bar{a}_m)_{12}\right] \\ -\mathrm{Im}\left[(\bar{a}_m)_{21}\right] & (\bar{a}_m)_{22} \end{bmatrix}$$

for liquid layers.

## NUMERICAL COMPUTATIONS

Computer programs were written in double precision Fortran IV to compute the pressure in dynes/cm$^2$ and the vertical particle displacement in millimicrons. The programs are run on the IBM 360/91. For numerical results presented here, computing time was under six minutes. It appears that practical limits of numerical precision make the present development most useful for the Arctic at frequencies below 50 Hz in water depths to 1 km. At higher frequencies or in greater water depths the integrands may be so oscillatory that it is difficult to achieve the desired accuracy in the numerical integrations without excessive computing time.

Computations are made in two stages. The first program, an extension of Dorman's (1962) PV 7 dispersion program, computes phase- and group-velocity dispersion, the excitation function dependent only on layering of the medium, and the excitation function for the particular source and detector depths. In the first case for the m-th normal mode the excitation function is defined by

$$\left\{ \frac{2\sqrt{2\pi}\sqrt{\omega}}{\rho_n c^{3/2} \bar{A}_{11} \dfrac{\partial}{\partial c}\left[\dfrac{c^2 \bar{A}_{11}}{\mathrm{Im}(r_{\alpha_n})} + \dfrac{\mathrm{Im}(-\bar{A}_{21})}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_m \\ c=c_m}}$$

for pressure and

$$\left\{ \frac{2\sqrt{2\pi}\sqrt{\omega}}{\rho_n c^{5/2} \bar{A}_{11} \frac{\partial}{\partial c}\left[\frac{c^2 \bar{A}_{11}}{Im(\Gamma_{\alpha_n})} + \frac{Im(-\bar{A}_{21})}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_m \\ c=c_m}}$$

for vertical particle velocity. In the second case the excitation function is

$$\left\{ \frac{2\sqrt{2\pi}\sqrt{\omega}\, Im\left[-(\bar{A}_{S_1})_{21}\right] Im\left[-(\bar{A}_{D_1})_{21}\right]}{\rho_n \rho_s c^{7/2} \bar{A}_{11} \frac{\partial}{\partial c}\left[\frac{c^2 \bar{A}_{11}}{Im(\Gamma_{\alpha_n})} + \frac{Im(-\bar{A}_{21})}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_m \\ c=c_m}}$$

for pressure and

$$\left\{ \frac{2\sqrt{2\pi}\sqrt{\omega}\, Im\left[-(\bar{A}_{S_1})_{21}\right]}{\rho_n \rho_s c^{5/2} \bar{A}_{11} \frac{\partial}{\partial c}\left[\frac{c^2 \bar{A}_{11}}{Im(\Gamma_{\alpha_n})} + \frac{Im(-\bar{A}_{21})}{\rho_n}\right]} \right\}_{\substack{\omega=\omega_m \\ c=c_m}}$$

for vertical particle velocity. These definitions were chosen to be useful also at long ranges where $H_0^{(2)}(kr) = \sqrt{\frac{2}{\pi k r}}\, e^{i(\pi/4 - kr)}$

The second program computes the three integrals, the normal mode contribution to the sound field, and writes and plots the absolute value of $p_{D_1}$, or $w_o$ as a

TABLE 1.

Model Parameters for Sorensen's (1959) Computations: Source Pressure Amplitude re 1m, .3048 dyne/cm$^2$; Source Depth, 15.2 m; Hydrophone Depth, 15.2m; Range, 3.048 m

| Layer Thickness, m | Compressional Velocity, m/sec | Shear Velocity, m/sec | Density gm/cm$^3$ |
|---|---|---|---|
| 18.3 | 1463.0 | 0 | 1.03 |
| Infinite | 1609.3 | 0 | 1.24 |

TABLE 2.

Integral Real Axis, Real Part

| Frequency, Hz | Sorensen's (1959) Value, dynes/cm$^2$ | Our Value, dynes/cm$^2$ |
|---|---|---|
| 10 | .004713 | .004685 |
| 20 | .030170 | .030153 |
| 40 | .040278 | .040485 |
| 80 | .023193 | .023121 |
| 160 | - .021839 | - .022201 |
| 320 | - .074395 | - .074011 |

TABLE 3.

Integral Real Axis, Imaginary Part

| Frequency, Hz | Sorensen's Value, dynes/cm$^2$ | Our Value, dynes/cm$^2$ |
|---|---|---|
| 10 | - .002567 | - .002551 |
| 20 | - .022260 | - .022240 |
| 40 | - .061960 | - .062274 |
| 80 | - .050986 | - .050641 |
| 160 | - .047979 | - .047750 |
| 320 | .006299 | .008316 |

TABLE 4.

Integral Imaginary Axis

| Frequency, Hz | Sorensen's Value, dynes/cm$^2$ | Our Value, dynes/cm$^2$ |
|---|---|---|
| 10 | .095080 | .095146 |
| 20 | .081938 | .082076 |
| 40 | .041887 | .041759 |
| 80 | .030795 | .030937 |
| 160 | .030214 | .030356 |
| 320 | .010593 | .011043 |

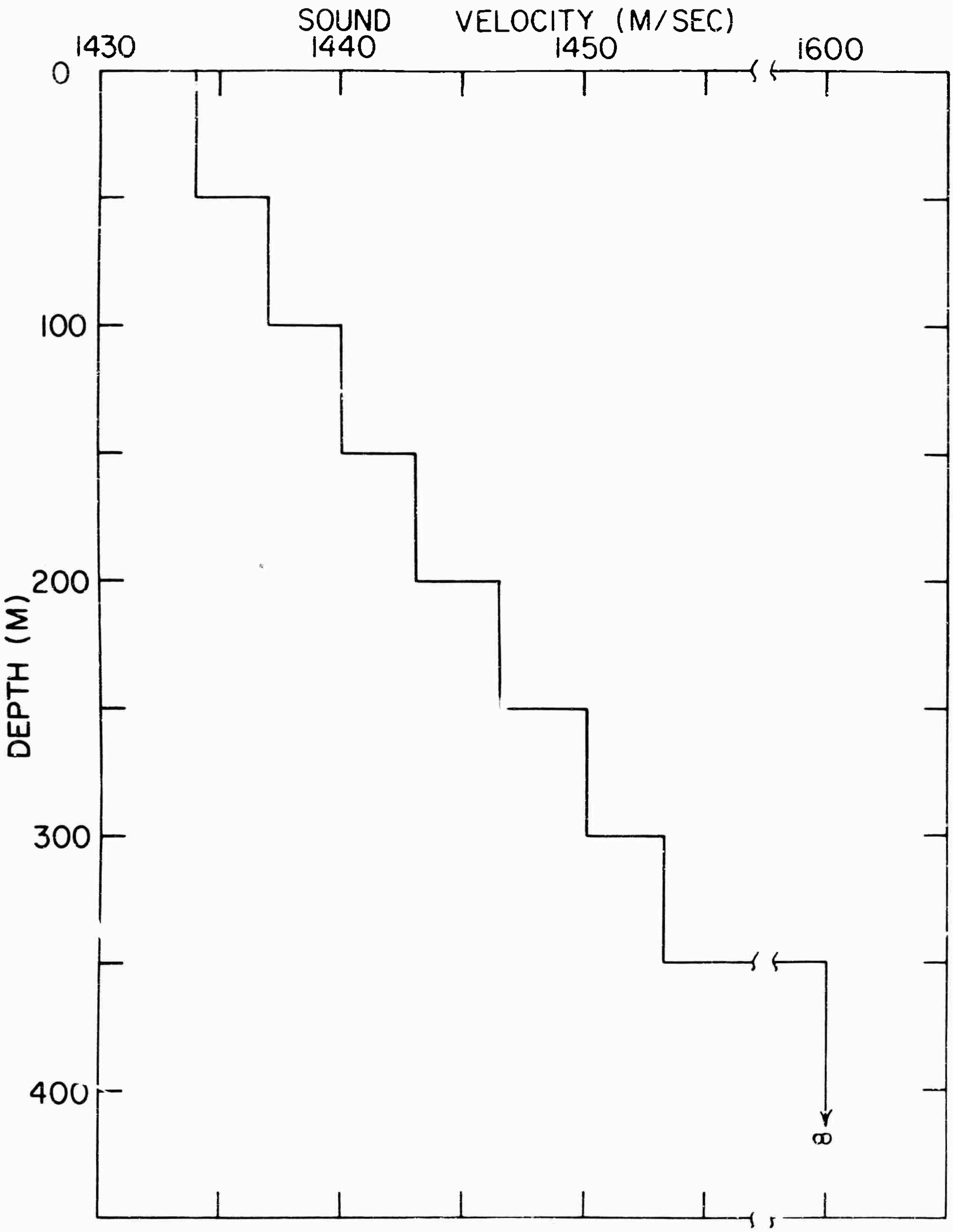
SOUND VELOCITY (M/SEC)
1430
1440
1450
1600
0
100
200
300
400
DEPTH (M)
∞

Fig. 3

TABLE 5.
Model A

| Layer Thickness, m | Compressional Velocity, m/sec | Shear Velocity, m/sec | Density, gm/cm$^3$ |
|---|---|---|---|
| 50 | 1434.0 | 0 | 1.03 |
| 50 | 1437.0 | 0 | 1.03 |
| 50 | 1440.0 | 0 | 1.03 |
| 50 | 1443.0 | 0 | 1.03 |
| 50 | 1446.0 | 0 | 1.03 |
| 50 | 1450.0 | 0 | 1.03 |
| 50 | 1453.2 | 0 | 1.03 |
| Infinite | 1600.0 | 0 | 1.20 |

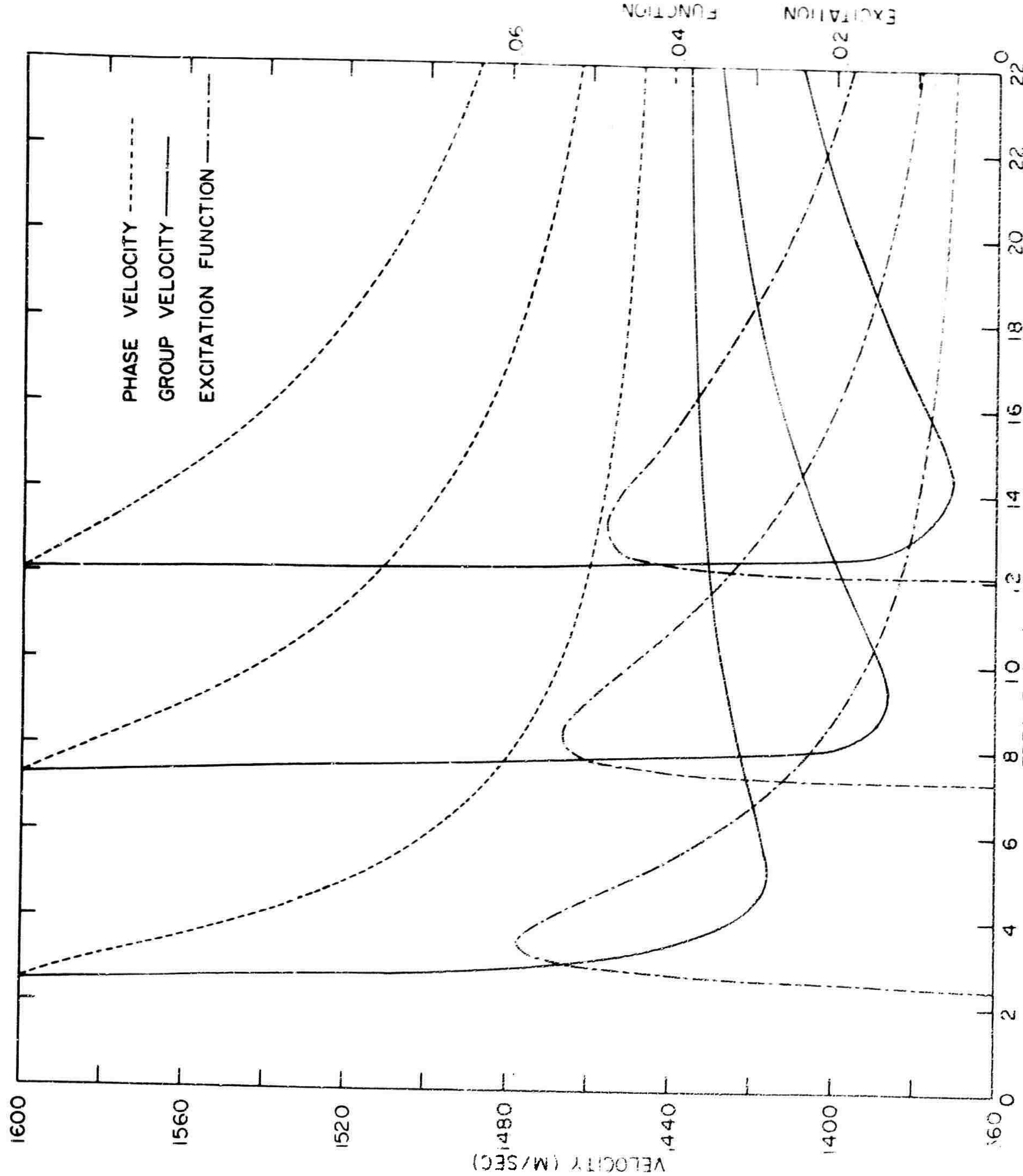
PHASE VELOCITY
GROUP VELOCITY
EXCITATION FUNCTION
VELOCITY (M/SEC)
1600
1560
1520
1480
1440
1400
1360
FREQUENCY (HZ)
0
2
4
6
8
10
12
14
16
18
20
22
EXCITATION FUNCTION
.06
.04
.02
0

Fig. 4

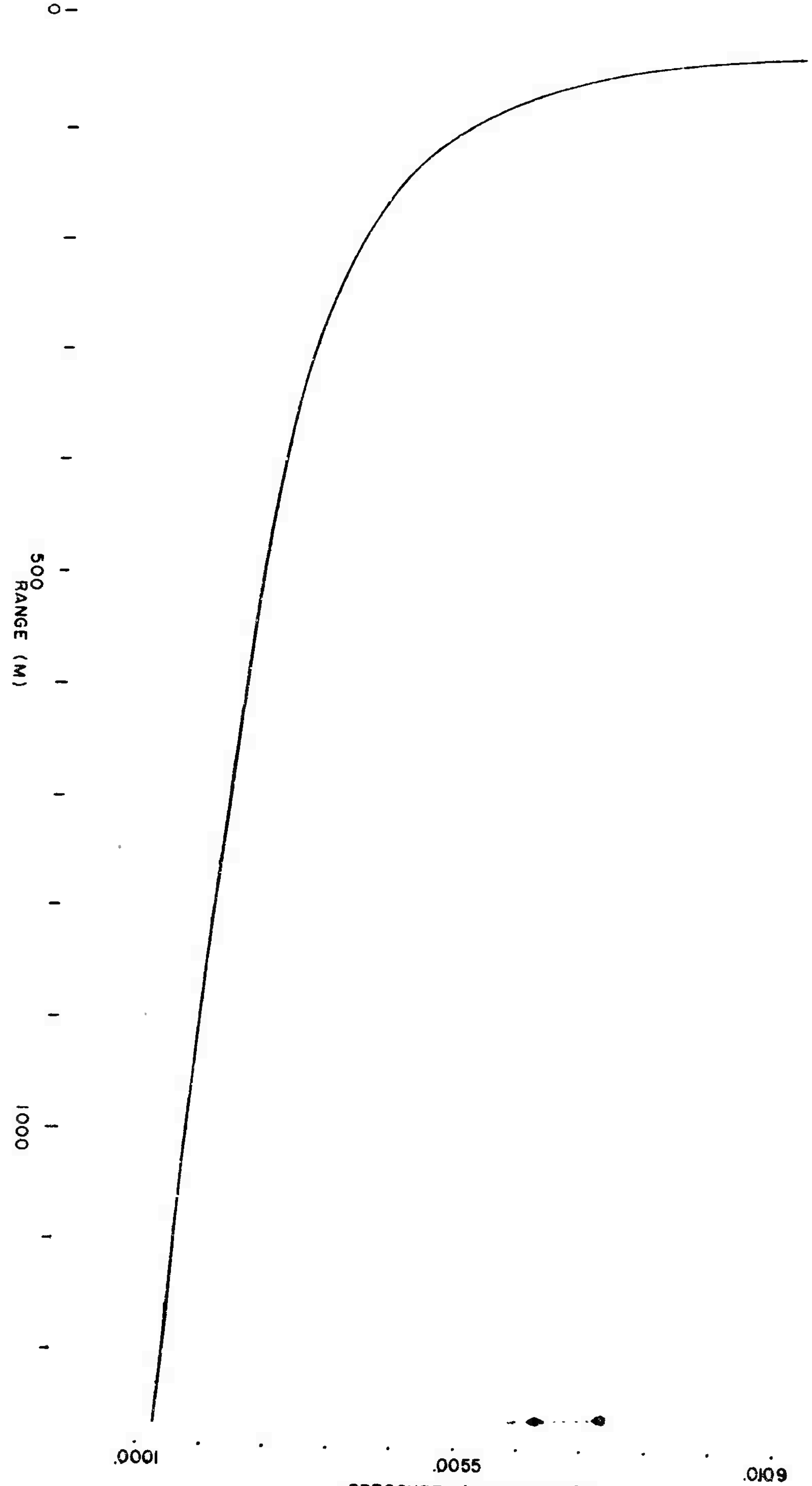


Fig. 5

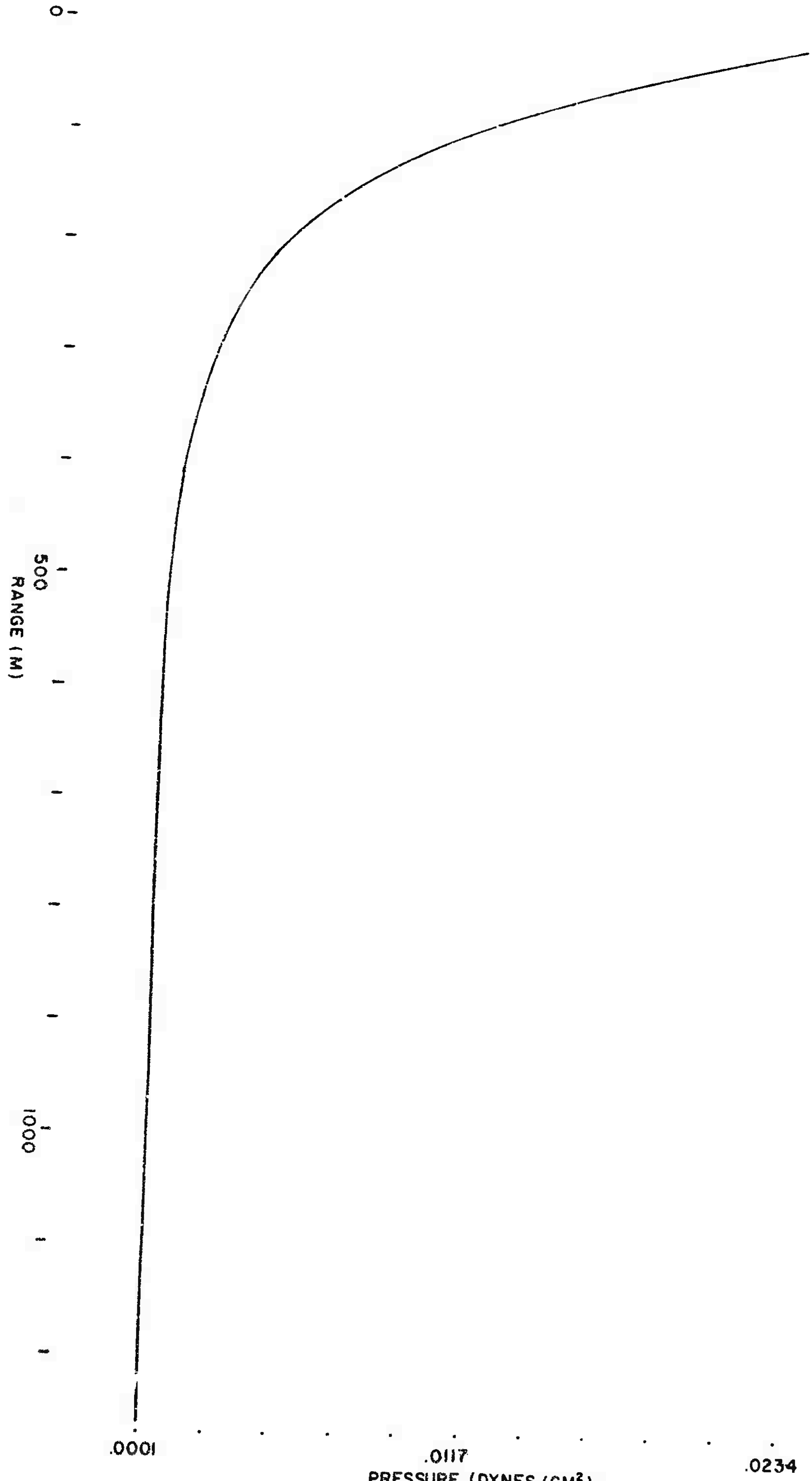


Fig. 6

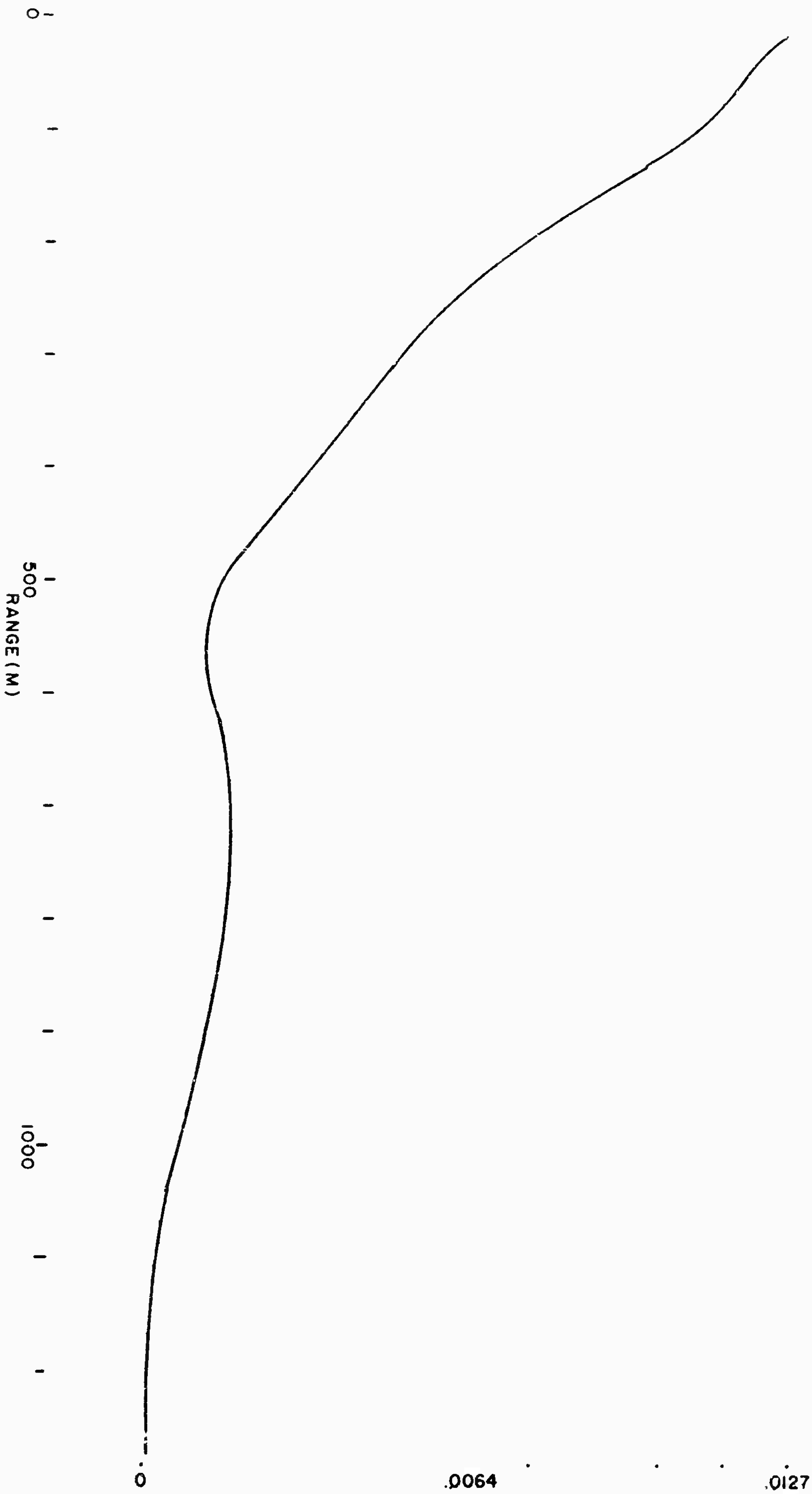


Fig. 7

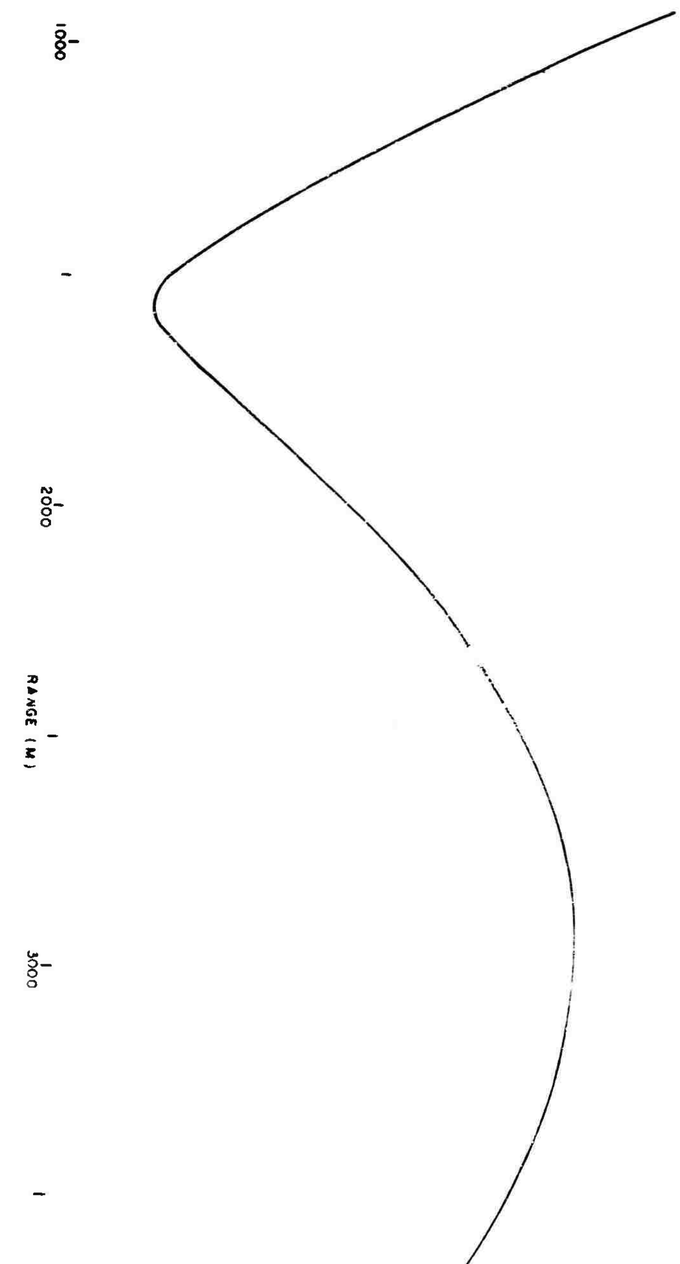


Fig. 8

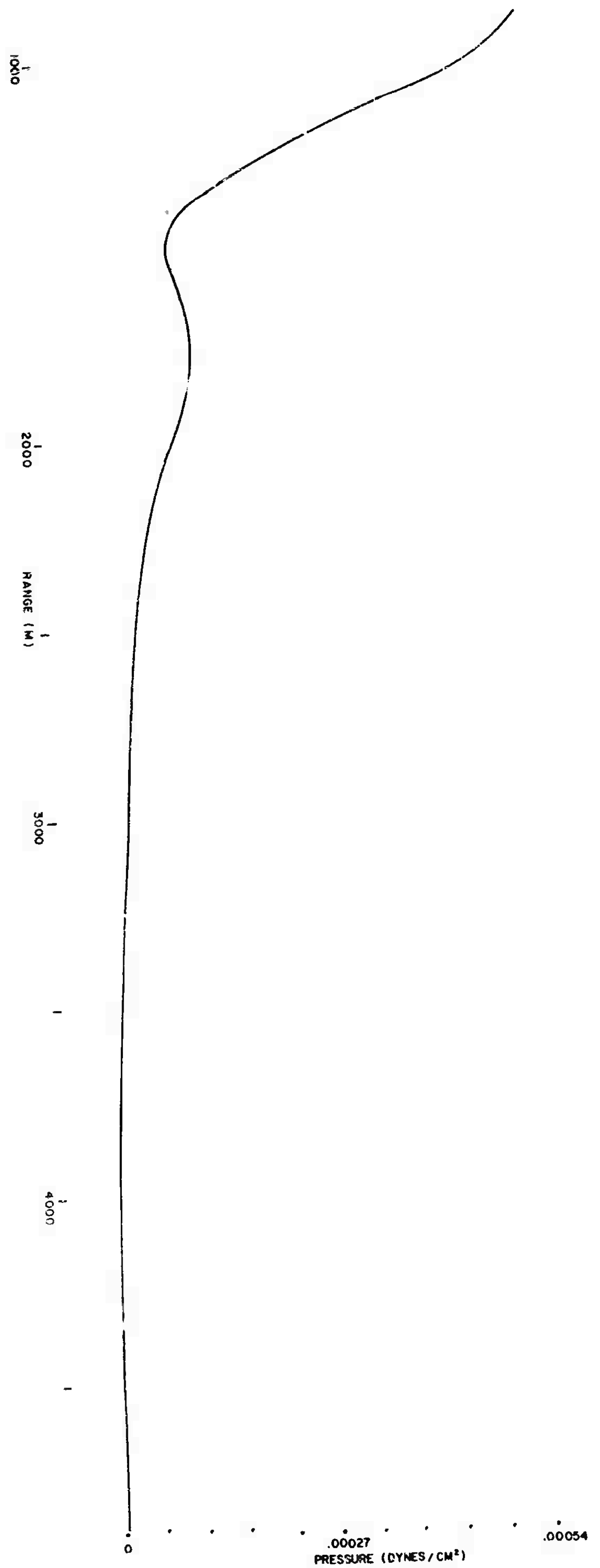


Fig. 9

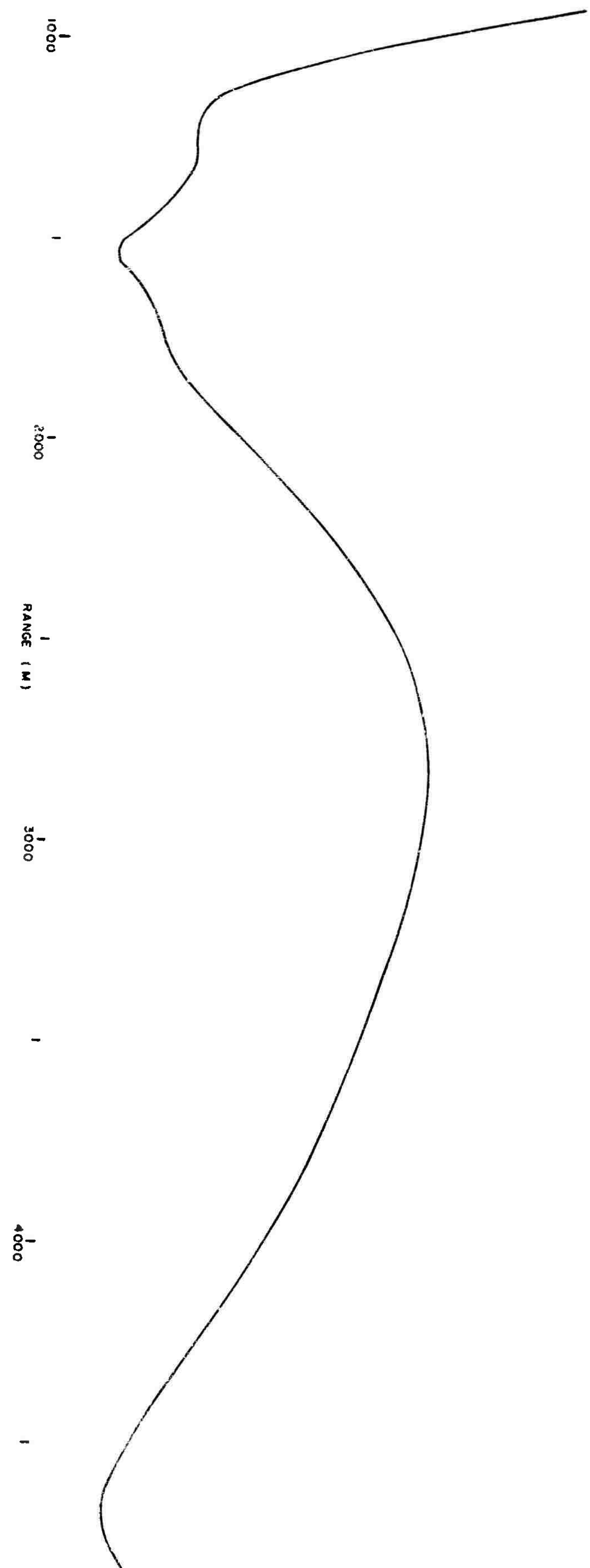


Fig. 10

function of range. In this program provision is also made to compute the three integrals as a function of frequency or detector depth at a specified range. The integrals are evaluated by Gaussian quadrature formulas (see, for example, Hildebrand, 1956). Care must be taken to employ a sufficient number of Gaussian points to obtain the desired accuracy of the integrals. A thirty-two point Gaussian quadrature formula was found to give sufficient accuracy for the computations presented here out to a range of 1500 m, but at longer ranges, higher frequencies, or greater water depths a ninety-six point formula is used. Abscissas and weight factors for the Gaussian integration are given by Davis and Rabinowitz (1956, 1958). The integration along the imaginary axis extends to infinity, but in practice a finite upper limit is chosen which gives sufficient accuracy. This may be done because the integrand is an exponential function of -kr for large k or r. The programs were checked by computations for the same two-layer model of Table 1 used by Sorensen (1959). Tables 2, 3 and 4 show that the two sets of numerical results are in very close agreement.

As an illustration of the progrms for the Arctic, computations were made for the layered model A of Figure 3, which closely follows the observed variation of sound velocity with depth. Additional parameters for the model are given in Table 5. This is a preliminary model and the ice sheet, which averages about 3-m in thickness in the central Arctic Ocean, was not included. The effects of the ice sheet on propagation are currently under investigation. Figure 4 shows phase - and group-velocity dispersion and the excitation function for pressure. At a frequency of 10 Hz only two normal modes are excited. The range dependence of the absolute

value of pressure for the normal mode contribution, the integral contribution, and the total pressure are shown in Figures 5 through 10.

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Dr. J. Dorman of the Lamont-Doherty Geological Observatory kindly supplied a copy of his PV-7 dispersion program. Computing facilities were provided by the Columbia University Computing Center. This work was supported by the U. S. Naval Ordnance Laboratory and the Office of Naval Research under contract N00014-67-A-0108-0016.

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# APPENDIX

Matrix elements for normal modes (see Haskell, 1953, and Dorman, 1962).

Solid layers

$$(\bar{a}_m)_{11} = (\bar{a}_m)_{44} = \gamma_m \cos \bar{P}_m - (\gamma_m - 1) \cos \bar{Q}_m$$

$$(\bar{a}_m)_{12} = (\bar{a}_m)_{34} = i\left[ (r_{\alpha_m}^{-1} (\gamma_m - 1)) \sin \bar{P}_m + \gamma_m r_{\beta_m} \sin \bar{Q}_m \right]$$

$$(\bar{a}_m)_{13} = (\bar{a}_m)_{24} = -(\rho_m c^2)^{-1} (\cos \bar{P}_m - \cos \bar{Q}_m)$$

$$(\bar{a}_m)_{14} = i(\rho_m c^2)^{-1} \left[ r_{\alpha_m}^{-1} \sin \bar{P}_m + r_{\beta_m} \sin \bar{Q}_m \right]$$

$$(\bar{a}_m)_{21} = (\bar{a}_m)_{43} = -i\left[ r_{\alpha_m} \gamma_m \sin \bar{P}_m + (r_{\beta_m}^{-1} (\gamma_m - 1)) \sin \bar{Q}_m \right]$$

$$(\bar{a}_m)_{22} = (\bar{a}_m)_{33} = -(\gamma_m - 1) \cos \bar{P}_m + \gamma_m \cos \bar{Q}_m$$

$$(\bar{a}_m)_{23} = i(\rho_m c^2)^{-1} (r_{\alpha_m} \sin \bar{P}_m + r_{\beta_m}^{-1} \sin \bar{Q}_m)$$

$$(\bar{a}_m)_{31} = (\bar{a}_m)_{42} = \rho_m c^2 \gamma_m(\gamma_m - 1)(\cos\bar{P}_m - \cos\bar{Q}_m)$$

$$(\bar{a}_m)_{32} = i\rho_m c^2\left[(r_{\alpha_m}^{-1}(\gamma_m - 1)^2)\sin\bar{P}_m + \gamma_m^2 r_{\beta_m}\sin\bar{Q}_m\right]$$

$$(\bar{a}_m)_{41} = i\rho_m c^2\left[\gamma_m^2 r_{\alpha_m}\sin\bar{P}_m + (r_{\beta_m}^{-1}(\gamma_m - 1)^2)\sin\bar{Q}_m\right]$$

Liquid layers

$$(\bar{a}_m)_{11} = (\bar{a}_m)_{44} = \cos\bar{P}_m$$

$$(\bar{a}_m)_{12} = \frac{i r_{\alpha_m}}{\rho_m c^2}\sin\bar{P}_m$$

$$(\bar{a}_m)_{21} = \frac{i\rho_m c^2}{r_{\alpha_m}}\sin\bar{P}_m$$

Matrix elements for integrals along real axis:

Solid layers

$$(a_m)_{11} = (a_m)_{44} = 2\left(\frac{k}{k_{\beta_m}}\right)^2\cos P_m - \left(2\left(\frac{k}{k_{\beta_m}}\right)^2 - 1\right)\cdot\cos Q_m$$

$$(a_m)_{12} = (a_m)_{34} = i\left[\frac{k\left(2\left(\frac{k}{k_{\beta_m}}\right)^2 - 1\right)\sin P_m}{\sqrt{k_{\alpha_m}^2 - k^2}} + \frac{2k}{k_{\beta_m}^2}\sqrt{k_{\beta_m}^2 - k^2}\,\sin Q_m\right]$$

$$(a_m)_{13} = (a_m)_{24} = -\frac{k}{\rho_m \omega}\left[\cos P_m - \cos Q_m\right]$$

$$(a_m)_{14} = i\left[\frac{k^2 \sin P_m}{\rho_m \omega \sqrt{k_{\alpha m}^2 - k^2}} + \frac{1}{\omega \rho_m}\sqrt{k_{\beta m}^2 - k^2}\,\sin Q_m\right]$$

$$(a_m)_{21} = (a_m)_{43} = -i\left[2\left(\frac{k}{k_{\beta m}}\right)^2 \frac{\sqrt{k_{\alpha m}^2 - k^2}}{k}\sin P_m\right.$$

$$\left. + \left(2\left(\frac{k}{k_{\beta m}}\right)^2 - 1\right)\frac{k \sin Q_m}{\sqrt{k_{\beta m}^2 - k^2}}\right]$$

$$(a_m)_{22} = (a_m)_{33} = \left[-\left(2\left(\frac{k}{k_{\beta m}}\right)^2 - 1\right)\cos P_m\right.$$

$$\left. + 2\left(\frac{k}{k_{\beta m}}\right)^2 \cos Q_m\right]$$

$$(a_m)_{23} = \frac{i}{\rho_m \omega}\left[\sqrt{k_{\alpha m}^2 - k^2}\,\sin P_m + \frac{k^2 \sin Q_m}{\sqrt{k_{\beta m}^2 - k^2}}\right]$$

$$(a_m)_{31} = (a_m)_{42} = \frac{2\rho_m \omega}{k}\left(\frac{k}{k_{\beta m}}\right)^2\left(2\left(\frac{k}{k_{\beta m}}\right)^2 - 1\right)\cdot$$

$$\left[\cos P_m - \cos Q_m\right]$$

$$(a_m)_{32} = \frac{i\rho_m \omega}{k}\left[\left(2\left(\frac{k}{k_{\beta m}}\right)^2 - 1\right)^2 \frac{k \sin P_m}{\sqrt{k_{\alpha m}^2 - k^2}} + \right.$$

$$2\left(\frac{k}{k_{\beta m}}\right)^2\Big)^2 \frac{\sqrt{k_{\beta m}^2-k^2}}{k^2} \sin Q_m$$

$$(a_m)_{41} = \frac{i\rho_m\omega}{k}\left[\left(2\left(\frac{k}{k_{\beta m}}\right)^2\right)^2 \frac{\sqrt{k_{\alpha m}^2-k^2}}{k}\sin P_m + \left(2\left(\frac{k}{k_{\beta m}}\right)^2-1\right)^2 \frac{k \sin Q_m}{\sqrt{k_{\beta m}^2-k^2}}\right.$$

Liquid layers

$$(a_m)_{11} = (a_m)_{44} = \cos P_m$$

$$(a_m)_{12} = \frac{i\sqrt{k_{\alpha m}^2-k^2}\sin P_m}{\omega\rho_m}$$

$$(a_m)_{21} = \frac{i\rho_m\omega}{\sqrt{k_{\alpha m}^2-k^2}}\sin P_m$$

Matrix elements for integral along imaginary axis:

Solid layers

$$P_m = h_m\sqrt{k_{\alpha m}^2+\tau^2}$$

$$Q_m = h_m\sqrt{k_{\beta m}^2+\tau^2}$$

$$(a_m)_{11} = (a_m)_{44} = -2\left(\frac{\tau}{k_{\beta m}}\right)^2\cos P_m + \left(2\left(\frac{\tau}{k_{\beta m}}\right)^2+1\right)\cdot \cos Q_m$$

$$(a_m)_{12} = (a_m)_{34} = \left(2\left(\frac{\tau}{k_{\beta m}}\right)^2+1\right)\frac{\tau\sin P_m}{\sqrt{k_{\alpha m}^2+\tau^2}} - \frac{2\tau}{k_{\beta m}^2}\sqrt{k_{\beta m}^2+\tau^2}\sin Q_m$$

$$(a_m)_{13} = (a_m)_{24} = \frac{-i\tau}{\rho_m\omega}\left[\cos P_m - \cos Q_m\right]$$

$$(a_m)_{14} = i\left[\frac{-\tau^2}{\rho_m\omega}\frac{\sin P_m}{\sqrt{k_{\alpha_m}^2+\tau^2}} + \frac{1}{\omega\rho_m}\sqrt{k_{\beta_m}^2+\tau^2}\sin Q_m\right]$$

$$(a_m)_{21} = (a_m)_{43} = \left[\frac{2\tau}{k_{\beta_m}^2}\sqrt{k_{\alpha_m}^2+\tau^2}\sin P_m - \left(2\left(\frac{\tau}{k_{\beta_m}}\right)^2+1\right)\frac{\tau\sin Q_m}{\sqrt{k_{\beta_m}^2+\tau^2}}\right]$$

$$(a_m)_{22} = (a_m)_{33} = \left(2\left(\frac{\tau}{k_{\beta_m}}\right)^2+1\right)\cos P_m - 2\left(\frac{\tau}{k_{\beta_m}}\right)^2\cos Q_m$$

$$(a_m)_{23} = \frac{i}{\rho_m\omega}\left[\sqrt{k_{\alpha_m}^2+\tau^2}\sin P_m - \frac{\tau^2\sin Q_m}{\sqrt{k_{\beta_m}^2+\tau^2}}\right]$$

$$(a_m)_{31} = (a_m)_{42} = i\left[-\frac{2\rho_m\omega\tau}{k_{\beta_m}^2}\left(2\left(\frac{\tau}{k_{\beta_m}}\right)^2+1\right)\cdot\cos P_m + \frac{2\rho_m\omega\tau}{k_{\beta_m}^2}\left(2\left(\frac{\tau}{k_{\beta_m}}\right)^2+1\right)\cos Q_m\right]$$

$$(a_m)_{32} = i\omega\rho_m \left[ \left( 2\left(\frac{\tau}{k_{\beta m}}\right)^2 + 1 \right)^2 \frac{\sin P_m}{\sqrt{k_{\alpha m}^2 + \tau^2}} - \left( \frac{2\tau}{k_{\beta m}} \right)^2 \sqrt{k_{\beta m}^2 + \tau^2} \sin Q_m \right]$$

$$(a_m)_{41} = i\omega\rho_m \left[ -\left( \frac{2\tau}{k_{\beta m}} \right)^2 \sqrt{k_{\alpha m}^2 + \tau^2} \sin P_m + \left( 2\left(\frac{\tau}{k_{\beta m}}\right)^2 + 1 \right)^2 \frac{\sin Q_m}{\sqrt{k_{\beta m}^2 + k^2}} \right]$$

Liquid layers

$$P_m = h_m \sqrt{k_{\alpha m}^2 + \tau^2}$$

$$(a_m)_{11} = (a_m)_{44} = \cos P_m$$

$$(a_m)_{12} = \frac{i\sqrt{k_{\alpha m}^2 + \tau^2} \sin P_m}{\omega\rho_m}$$

$$(a_m)_{21} = \frac{i\rho_m \omega \sin P_m}{\sqrt{k_{\alpha m}^2 + \tau^2}}$$

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13. ABSTRACT

This report develops by matrix methods the integral solution of the wave equation for point sources of harmonic waves in a liquid layer of a multilayered liquid-solid half space in a form convenient for numerical computation on a high-speed digital computer. Only the case is considered of a high-speed liquid bottom underlying the stack of layers. The integral over wave number has singularities in the integrand and is conveniently transformed into the complex plane. By a proper choice of contours, complex poles are displaced to an unused sheet of the two-leaved Riemann surface, and the integral solution for the multilayered system reduces to a sum of normal modes plus the sum of two integrals, one along the real axis and the other along the imaginary axis. Both integrals are evaluated by a Gaussian quadrature formula. Sample computations are presented for low-frequency propagation in the Arctic Ocean sound channel. These are preliminary computations and the ice layer, which averages three meters in thickness, is not included in the layered system. The effects of the ice layer on propagation are currently under investigation.

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| Arctic | | | | | | |
| Hydroacoustics | | | | | | |
| Sound field | | | | | | |
| Layered media | | | | | | |
| Matrix | | | | | | |
| Integral solution | | | | | | |
| Digital computer | | | | | | |
| Numerical computations | | | | | | |

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